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A Shooting Approach to Layers and Chaos in a Forced Duffing Equation

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We study equilibrium solutions for the problem

$$u_t = \varepsilon^2 u_{xx} - u^3 + \lambda u + \cos x,$$
$$u_x(0, t) = u_x(1, t) = 0.$$

Using a shooting method we find solutions for all nonzero ε. For small ε we add to the solutions found by previous authors, especially Angenent, Mallet-Paret and Peletier, and Hale and Sakamoto, and also give new elementary ode proofs of their results. Among the new results is the existence of internal layer-type solutions. Considering the ode satisfied by equilibria, but on an infinite interval, we obtain chaos results for $\lambda \geqslant \lambda_0 = \frac{3}{2^{2/3}}$ and $0 < \varepsilon \leqslant \frac{1}{4}$. We also consider the problem of bifurcation of solutions as λ increases. © 2002 Elsevier Science (USA)

Key Words: Duffing equation; layer solutions; shooting methods; chaos.

1. INTRODUCTION

In this paper, we begin a study of the bounded solutions of equations of the form

$$\varepsilon^2 u'' = u^3 - \lambda u + g(t), \tag{1.1}$$

where λ and ε are positive parameters and where g is a smooth periodic function. In much of the paper $g(t) = \cos t$, and the equation is a standard model in the theory of nonlinear oscillations, one of several which have been called a forced Duffing equation in the literature. With the signs shown it is often referred to as the equation of a "soft spring." A comprehensive reference to the early theory of (1.1) is [NM], which gives a detailed account of the results obtained by classical perturbation methods, such as averaging

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or multiscale techniques. More recent efforts have used dynamical systems concepts to establish results about chaotic behavior of one sort or another.

We obtain some preliminary results for a general bounded function g. The choice of $g = \cos t$ for most of the work enables us to use symmetry arguments which shorten several proofs. This choice also simplifies the behavior in some respects, and we discuss this in Section 5.1.

An important reference is by Angenent, Mallet-Paret, and Peletier [AMPP], who studied stable steady states for a reaction–diffusion equation

$$u_\tau = \varepsilon^2 u_{xx} + f(x, u) \tag{1.2}$$

with boundary conditions

$$u_x(0, \tau) = u_x(L, \tau) = 0 \tag{1.3}$$

for a class of functions f which were cubic in the state variable u. Equation (1.1) is obtained from (1.2) (with $f(x, u) = -u^3 + \lambda u - g(x)$) by setting $x = t$ and assuming that u is independent of τ. While the specific form (1.1) is not mentioned in [AMPP], the methods there are easily seen to apply and to prove important results about the existence of stable steady states of the problem (1.2)–(1.3) with this form of the function f and for sufficiently small ε. Compactness requirements, however, seem to make it more difficult to study the problem on an infinite interval, and thereby get "chaos," using infinite dimensional methods.

In this paper, we introduce a shooting technique, which we have not seen in this form elsewhere, which is the basis of our approach. We use this technique to obtain new periodic solutions over a range of ε which is not "small." As far as we know, this is the first method to yield results on this problem for a specific interval of parameter values. We show that one can give rigorous results about a weak form of "chaos" over this larger range of ε. We also introduce the problem of determining how these solutions arise as λ varies, since for $\lambda \leqslant 0$, (1.1) will be shown to have a unique bounded solution. Further work on this problem is in progress. Then, letting ε return to its traditional role as a small parameter, we reproduce and extend the results in [AMPP] as they apply to this equation, using elementary ode methods. We study the bifurcation problem in λ in more detail, and we begin a study of a further class of bounded solutions which we can find for small ε.

In most of this paper we will consider the ode

$$\varepsilon^2 u'' = u^3 - \lambda u + \cos t \tag{1.4}$$

for its own sake. Some of the solutions found by our methods are periodic and some are "chaotic" in a sense to be defined below. Those which are periodic are unstable as steady states of (1.2)–(1.3) and therefore apparently not found by the pde techniques of [AMPP] . In [HS], Hale and Sakamoto did discover some unstable solutions for small ε, using complete different methods from ours. Recent work of Nakashima on a similar problem uses continuation methods which may yield results similar to some of ours for this problem, such as the existence of these new periodic solutions [NAK]. There have been many other papers on the problem (1.2)–(1.3) with cubic nonlinearities, and some of these also consider unstable solutions. Two recent ones, which cite other related work, are by Hale and Salazar [HS1, HS2]. An earlier paper by Kurland gives many oscillatory solutions which are similar to some of those we find in Section 4.2, but for a different class of nonlinear functions $f(x,u)$ [KUR]. The use of formal asymptotic analysis on similar problems has been studied by Ockendon, Ockendon and Johnson [OOJ], Norbury and Yeh [NY] and Mays and Norbury [MN].

All of these papers use methods which appear to be quite different from ours, and study, for the most part, equations different from (1.1). We have not investigated whether these methods, many using infinite dimensional functional analysis or sophisticated topology, apply to (1.1). Most do not mention equations of the general form of ours (as in (2.1)), [AMPP] being a notable exception. Most do not study problems on an infinite interval, so that chaos is not considered. We have seen no other work on the bifurcation problem considered in Section 4.5.

The proof of a weak form of chaos which we give seems to us to be very simple. No analysis is required beyond a simple phase plane argument and the continuity of solutions with respect to initial conditions. See Theorem 3.2 and also the first part of Section 3.4. This is for small ε. For larger ε, a few estimates are needed to verify the hypotheses of Theorem 3.2. These also appear in Section 3.4.

To relate our results to those of traditional nonlinear oscillation theory, we recall that in [NM] the undamped form of the equation is written as

$$\ddot{u} + \omega_0^2 u = \varepsilon(u^3 + k\cos((w_0 + \varepsilon\sigma)t)). \tag{1.5}$$

This form is chosen to study "near-resonance" phenomena, which were the main interest of much previous work. Resonance, or near-resonance, will play no role in our approach, since we are not studying a small perturbation of a linear problem. A rescaling of (1.5) to put the equation in the form (1.1) shows that ε small in (1.5) corresponds to λ large in (1.1), though the analogy is not exact because this rescaling also results in a small-amplitude high-frequency forcing.

The paper is organized partly according to the restrictions place on ε. The results in Section 2 are for any $\varepsilon > 0$. The results in Section 3 are for a range of ε which can be stated explicitly, while the results in Section 4 are for "sufficiently small" ε. A more detailed outline of the paper can be found in Section 5. Figures 2–7 were generated by the software package xpp [ERM].

2. RESULTS FOR ALL POSITIVE ε

At this initial stage it is easy, and we believe of some interest, to study a more general class of equations. So to start with, consider the equation

$$\varepsilon^2 u'' = u^3 - \lambda u + g(t), \tag{2.1}$$

where g is any continuous bounded function on $[0, \infty)$, with initial conditions

$$u(0) = \alpha, \qquad u'(0) = 0. \tag{2.2}$$

We denote the unique solution by u_α. Our initial interest for a general g is to find solutions which are bounded on $[0, \infty)$. Fixing $\varepsilon > 0$, this can be viewed as a bifurcation problem in λ. There is a "main branch" of bounded solutions which exist for all λ, as given in the following result.

PROPOSITION 2.1. *For any* $\varepsilon > 0$ *and any* λ, (2.1) *has a solution satisfying* (2.2) *which is bounded on* $[0, \infty)$.

Proof. Choose $b > 0$ so large that if $|u| \geqslant b$ then $u^3 - \lambda u + g(t) \neq 0$, for any t. Then, $u'' > 0$ when $u \geqslant b$ and $u'' < 0$ when $u \leqslant -b$. We define two subsets of the α axis:

$$A = \{\alpha \mid u_\alpha(x) > b \text{ for some } x \geqslant 0\},$$
$$B = \{\alpha \mid u_\alpha(x) < -b \text{ for some } x \geqslant 0\}.$$

These sets are clearly nonempty (e.g. $b + 1 \in A$) and open. They are disjoint because u_α cannot have a local maximum in the region $u > b$ or a local minimum in the region $u < -b$. Hence, there is an $\alpha^* \notin A \cup B$, and this corresponds to a solution which is bounded on $[0, \infty)$. This proves Proposition 2.1. ∎

It should be remarked that for any even function $g(\cdot)$, such as cosine, the solutions u_α are even and so u_{α^*} is bounded on the whole real line.

PROPOSITION 2.2. *For any* $\lambda \leqslant 0$ *there is only one solution of* (2.1)–(2.2) *which is bounded on* $[0, \infty)$.

Proof. Suppose that there are two bounded solutions, say u_1 and u_2, and let $v = u_1 - u_2$. Then

$$\varepsilon^2 v'' = (u_1^2 + u_1 u_2 + u_2^2 - \lambda)v, \qquad v'(0) = 0,$$

and since $u_1^2 + u_1 u_2 + u_2^2$ is positive definite and $\lambda \leqslant 0$, v cannot be bounded on $[0, \infty)$. This proves the result. ∎

Remark. 2.3. A similar argument shows that for a general continuous bounded function g, when $\lambda \leqslant 0$ there is exactly one solution of (2.1) which is bounded on $(-\infty, \infty)$. However the existence part of this result is a little longer, requiring a two-parameter "shooting" argument and some more complicated topology.

We now consider what happens as λ increases from zero. Consider the function $p_\lambda(u) = u^3 - \lambda u$. As λ crosses zero, p develops, by a "pitchfork bifurcation," two new roots (besides $u = 0$), a local maximum at $u = -\sqrt{\frac{\lambda}{3}}$ and a local minimum at $u = \sqrt{\frac{\lambda}{3}}$. As λ increases further, the value of p_λ at these maxima and minima will eventually exceed the maximum of $|g(t)|$. Let

$$\lambda_0 = \sup\left\{ \lambda \mid |g(t)| \geqslant p_\lambda\left(-\sqrt{\frac{\lambda}{3}}\right) \text{ for some } t \right\}.$$

PROPOSITION 2.4. *If* $\lambda > \lambda_0$, *then for any* $\varepsilon > 0$ *there is a unique bounded solution with* $u > 0$, $3u^2 - \lambda > 0$, *and another bounded solution, also unique, with* $u < 0$, $3u^2 - \lambda > 0$.

Proof. Existence of these solutions is a simple application of the shooting technique from Proposition 2.1. For the positive solution we consider initial conditions $u(0) = \alpha \in (\sqrt{\frac{\lambda}{3}}, b)$, where b is defined in the proof of Proposition 2.1. We again define two sets:

$$A = \left\{ \alpha \in \left(\sqrt{\frac{\lambda}{3}}, b\right) \middle| u_\alpha(t) > b \text{ for some } t > 0 \right\}$$

and

$$B = \left\{ \alpha \in \left(\sqrt{\frac{\lambda}{3}}, b\right) \middle| u_\alpha(t) < \sqrt{\frac{\lambda}{3}} \text{ for some } t > 0 \right\}.$$

Because $u'' < 0$ if $u = \sqrt{\frac{\lambda}{3}}$ and $u'' > 0$ if $u = b$, both of these sets are open and nonempty, and as in the proof of Proposition 2.1 there is an α^* in $(\sqrt{\frac{\lambda}{3}}, b)$

which is not in $A \cup B$. The solution $u_{\alpha*}$ has the properties described in the lemma. A similar proof gives the negative solution described in the lemma. Uniqueness of both of these solutions follows as in Proposition 2.2. ∎

Remark. 2.5. We believe that there should be a third bounded solution as well, but we do not have a proof for a general continuous bounded function g. There appears to be a significant difference between boundary value problems on finite intervals, such as those in [AMPP] , and the semi-infinite interval problem discussed here. It is easy to show, for a bounded g and any given $L > 0$, that for large enough λ there are at least three solutions of (2.1)–(2.2) such that $u'(L) = 0$. But it does not appear to us to be so easy to find three bounded solutions on $[0, \infty)$. "Generically" there should be three, perhaps, by some sort of degree theory argument. However, the noncompactness of $[0, \infty)$ seems to make the use of such arguments more challenging, especially if the goal is to remove the qualification "generically."

So now we specialize to the equation of principle interest in this paper, namely (1.4). It is useful to consider the curve in the (t, u) plane defined implicitly by the equation $u'' = 0$; that is, by $f(t, u) \equiv u^3 - \lambda u + \cos t = 0$. This curve has one component for $\lambda \leqslant \lambda_0 = \frac{3}{2^{2/3}}$ and three components for $\lambda > \lambda_0$. The graphs for $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$, and $\lambda > \lambda_0$ are shown in Fig. 1.

In all three cases, the curve crosses $u = 0$ at odd multiples of $\frac{\pi}{2}$. X. Chen has observed that at $\lambda = \lambda_0$ there is an exact formula for the lower curve over $[0, \pi]$, namely, $-2^{2/3} \cos\left(\frac{t}{3}\right)$. There are corresponding formulas for the other parts of the curve for this value of λ.

For $\lambda > \lambda_0$, let $\bar{U}(t) > U_0(t) > \underline{U}(t)$ be the three solutions of $u^3 - \lambda u + \cos(t) = 0$, which are all 2π-periodic. Note that $\underline{U}(t) = -\bar{U}(t + \pi)$.

Theorem 2.6. *When* $g(t) = \cos t$, *let* u_1 *and* u_5 *be the solutions found in Proposition* 2.4, *with* $u_1 < 0$ *and* $u_5 > 0$. *Then these solutions are* 2π *periodic,*

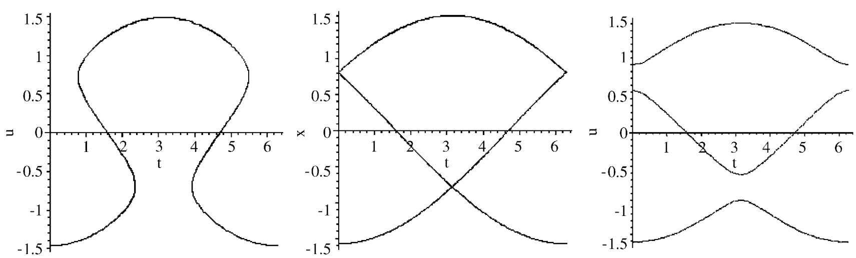

FIGURE 1

$u_1' > 0$ in $(0,\pi)$ and $\underline{U}(0) < u_1 < \underline{U}(\pi)$ in $[0,\pi]$. Also, $u_5(t) = -u_1(t+\pi)$ and u_1 and u_5 are the minimal and maximal bounded solutions of (1.4) *on* $(-\infty,\infty)$, *in the sense that every other bounded solution lies between these two. Also there is a third 2π-periodic solution, $u_p(= u_{\alpha_p})$, which satisfies $u\left(\frac{\pi}{2}\right) = 0$ and $u' > 0$ on $\left(0,\frac{\pi}{2}\right)$. A unique solution with these properties exists for all λ (and so for $\lambda \leqslant 0$ this is the solution found in Proposition* 2.1).

Proof. In proving that the solutions of Proposition 2.4 are periodic, we give a second proof of their existence, for the case $g(t) = \cos t$. Let $\alpha_0 = \underline{U}(0)$. First, observe that $u''_{\alpha_0}(0) = u'''_{\alpha_0}(0) = 0$ while $\varepsilon^2 u^{(4)}_{\alpha_0}(0) = -1$. Therefore, $u_{\alpha_0}(t)$ decreases monotonically to $-\infty$. If $\alpha_0 < \alpha < U_0(0)$, then $u''_\alpha(0) > 0$ and so u_α initially increases. However, for sufficiently small positive values of $\alpha - \alpha_0$, u'_α has a first zero, which we denote by $t_1(\alpha)$. This means, in turn, that u''_α must have a first zero at $\tau_1(\alpha) \in (0, t_1(\alpha))$. For small $\alpha - \alpha_0 > 0$, $t_1(\alpha) < \frac{\pi}{2}$. Further, $\varepsilon^2 u'''_\alpha(\tau_1(\alpha)) \leqslant 0$. From

$$\varepsilon^2 u''''(t) = (3u^2 - \lambda)u'' + 6uu'^2 - \cos t$$

we conclude that for small $\alpha - \alpha_0 > 0$, $u''_\alpha < 0$ on $(\tau_1(\alpha), t_1(\alpha)]$. Hence, $u(t_1(\alpha)) < \underline{U}(t_1(\alpha))$.

Clearly, $t_1(\alpha)$ approaches 0 as α tends to α_0 from above. Extend the function $t_1(\cdot)$ to larger α by continuity, as long as possible. That is, $t_1(\cdot)$ is the continuous function such that $u'_\alpha(t_1(\alpha)) = 0$ and, for α sufficiently close to α_0, $t_1(\alpha)$ is the first positive zero of u'_α. Then $t_1(\alpha)$ remains the first zero of u_α until either (i) $u''_\alpha(t_1(\alpha)) = 0$ (since we can then no longer use the implicit function theorem to solve $u'_\alpha(t) = 0$ for t) or (ii) there exists a first $t_0 \in (0, t_1(\alpha))$ such that $u'_\alpha(t_0) = 0$ and $u''_\alpha(t_0) = 0$ (since then $t_1(\alpha)$ ceases to be the first positive zero of u'_α). But if (i) occurs at some first $\alpha_1 > \alpha_0$, and $t_1(\alpha_1)$ is the first zero of u'_{α_1}, with $t_1(\alpha_1) < \pi$, then $u'''_{\alpha_1}(t_1(\alpha_1)) = -\sin t_1(\alpha_1) < 0$. This implies that $u'_{\alpha_1} = 0$ to the left of $t_1(\alpha_1)$, a contradiction. If (ii) occurs we get the same contradiction at $t_0(\alpha)$. If (i) occurs at α_1 and $t_1(\alpha_1) = \pi$, then $u'''_{\alpha_1}(t_1(\alpha_1)) = 0$ and $\varepsilon^2 u''''_{\alpha_1}(t_1(\alpha_1)) = 1$. This means that $u''_{\alpha_1} > 0$ on either side of $t_1(\alpha_1) = \pi$, and $t_1(\alpha_1)$ could not be the first zero of u'_{α_1}. Hence $t_1(\alpha)$ is continuous and remains the first zero of u'_α as long as $t_1(\alpha) \leqslant \pi$.

On the other hand, if $\alpha = \underline{U}(\pi)$, then the first maximum of u_α is larger than $U_0(\pi)$. Therefore, $t_1(\alpha)$ must vary continuously as α increases from α_0 until it eventually takes on the value π for some $\alpha \in (\alpha_0, \underline{U}(\pi))$. For any such α, u_α gives a periodic solution of (1.4) with the properties that $u'_\alpha(t) > 0$ for $t \in (0,\pi)$ and $\underline{U}(0) < u_\alpha(t) < \underline{U}(\pi)$ for $t \in [0,\pi]$. From Proposition 2.4, it follows that such an α is unique and the corresponding solution u_α coincides with the unique bounded solution with $u < 0$ and $3u^2 - \lambda < 0$ in Proposition 2.4. The reflection $-u_\alpha(\pi + t)$ of this periodic solution gives a second periodic solution, which corresponds to the unique bounded solution with $u > 0$ and $3u^2 - \lambda > 0$ in Proposition 2.4.

The proof that u_1 and u_5 are the minimal and maximal bounded solutions of (1.4) on $(-\infty, \infty)$ follows from that of Proposition 2.4.

To obtain the third periodic solution, u_{p}, again let $\alpha_0 = \underline{U}(0)$, the smallest root of $u^3 - \lambda u + 1 = 0$. We saw that if $\alpha < \alpha_0$ then $u^3 - \lambda u + \cos t < 0$ for all $u \leqslant \alpha$, and all t, and u_α decreases monotonically to $-\infty$, (in finite time). If $\alpha = \alpha_0$, then from the above we know that u_α decreases below α_0 and tends to $-\infty$. On the other hand, if $\alpha < 0$ is sufficiently small in magnitude then $\varepsilon^2 u_\alpha''(0) > \frac{1}{2}$ and it is easy to show that u_α increases to cross zero at some first $t_0 = t_0(\alpha)$, with $u_\alpha' > 0$ on $(0, t_0(\alpha)]$. Lowering α from 0, the function $t_0(\alpha)$ is continuous as long as $u_\alpha'(t_0(\alpha)) > 0$. If $u_\alpha'(t_0(\alpha)) = 0$ then $u_\alpha''(t_0(\alpha)) \leqslant 0$ since u_α cannot have a local minimum at $t_0(\alpha)$. This implies that $t_0(\alpha) \geqslant \frac{\pi}{2}$. Since $t_0(\alpha)$ is not defined for $\alpha \leqslant \alpha_0$, it must increase beyond $\frac{\pi}{2}$ as α decreases from zero. Thus, there is some first (largest) $\alpha = \alpha_{\mathrm{p}} < 0$ with $t_0(\alpha_{\mathrm{p}}) = \frac{\pi}{2}$. Let $u_{\mathrm{p}} = u_{\alpha_{\mathrm{p}}}$. An easy symmetry argument shows that $u_{\mathrm{p}}\left(\frac{\pi}{2} + t\right) = -u_{\mathrm{p}}\left(\frac{\pi}{2} - t\right)$ for all t, showing that u_{p} is 2π-periodic, with a maximum at π.

Suppose, therefore, that $u = u_{\mathrm{p}}$ satisfies $u'(\tau) = 0$ for some first $\tau \in (0, \pi)$. By the anti-symmetry of u_{p} around $\frac{\pi}{2}$ we can assume that $\tau \leqslant \frac{\pi}{2}$. Then $u''(\tau) \leqslant 0$. If $u''(\tau) = 0$ then $\varepsilon^2 u'''(\tau) = -\sin\tau < 0$ so u'' becomes negative. In either case, $u' < 0$ on some interval to the right of τ.

From the graphs of $u'' = 0$ it is clear that if $u(0) < 0$, $u''(0) > 0$, $u < 0$ on $\left(0, \frac{\pi}{2}\right)$ and $u' = 0$ before $t = \frac{\pi}{2}$, then after that point, $u'' < 0$ at least until $t = \pi$. This implies that $u_{\mathrm{p}}' > 0$ on $(0, \pi)$.

The uniqueness of u_{p} follows from Lemma 2.7 below.

LEMMA 2.7. *Assume that* $\alpha_1 < \alpha_2 < 0$. *If* $0 < T \leqslant \pi/2$ *and* $u_{\alpha_2} \leqslant 0$ *in* $[0, T]$, *then* $u_{\alpha_1} < u_{\alpha_2}$ *in* $[0, T]$.

Proof. First, using the Sturm–Liouville technique we get

$$\begin{aligned} &\varepsilon^2 (u_{\alpha_1}' u_{\alpha_2} - u_{\alpha_1} u_{\alpha_2}')(t) \\ &\quad = \int_0^t u_{\alpha_1} u_{\alpha_2} (u_{\alpha_1}^2 - u_{\alpha_2}^2)\, ds + \int_0^t (u_{\alpha_2} - u_{\alpha_1}) \cos t\, ds. \end{aligned} \tag{2.3}$$

Assume that the lemma is false. Then, there is a first $t = \bar{t} \in (0, \pi/2]$ such that $u_{\alpha_2} \leqslant 0$ in $[0, \bar{t}]$ and $u_{\alpha_1} = u_{\alpha_2}$ at $\bar{t}$. Then, $u_{\alpha_1}'(\bar{t}) \geqslant u_{\alpha_2}'(\bar{t})$. Evaluating (2.3) at $t = \bar{t}$ shows that the left side is nonpositive and the right side is positive. This contradiction proves the lemma and completes the proof of Theorem 2.6. ∎

Below we will want a slight variant of Lemma 2.7, with a similar proof which we omit.

LEMMA 2.8. *Let u and U be two solutions of* (1.4) *with* $u(\pi) < U(\pi)$ *and* $u'(\pi) = U'(\pi) = 0$. *If* $\frac{\pi}{2} \leqslant T < \pi$ *and* $u \geqslant 0$ *on* $[T, \pi]$, *then* $u < U$ *in* $[T, \pi]$.

COROLLARY 2.9. *Assume that u is a periodic solution of* (1.4) *with* $u'(0) = u'(\pi) = 0$. *If* $u(0) < u_p(0)$, *then* $u(t) < u_p(t)$ *for all* $t \in (-\infty, \infty)$.

Proof. From Lemma 2.7 it suffices to show that $u(t) < u_p(t)$ for $t \in (\pi/2, \pi]$. If $u(\pi) > u_p(\pi)$, then Lemma 2.8 implies $u(t) > u_p(t)$ for all $t \in \left[\frac{\pi}{2}, \pi\right]$ so that u has a jump at $\frac{\pi}{2}$, a contradiction. Hence, we have $u(\pi) < u_p(\pi)$. Assume now that the corollary is false. Since $u_p' > 0$ in $(0, \pi)$, there is a $\bar{t} \in \left(\frac{\pi}{2}, \pi\right)$ such that $u'(\bar{t}) = 0$ and $u(\bar{t}) > u_p(\bar{t})$. The Sturm–Liouville technique shows that

$$\begin{aligned} &\varepsilon^2[u'(t)u_p(t) - u(t)u_p'(t)] \\ &\quad = -u(\bar{t})u_p'(\bar{t}) + \int_{\bar{t}}^{t} [uu_p(u^2 - u_p^2) + (u_p - u)\cos s]\, ds. \end{aligned}$$

If $u(t) = u_p(t)$ for some largest $t \in \left[\frac{\pi}{2}, \bar{t}\right]$ then $\left(\frac{u}{u_p}\right)' < 0$ in $(t, \bar{t})$ a contradiction. We again get that u has a jump at $t = \frac{\pi}{2}$ and this proves the corollary. ∎

We have now shown that as λ increases from 0, new periodic solutions appear. We can prove one thing about the initial bifurcation of new solutions for any positive ε. Let

$$\lambda_b = \sup\{\lambda \mid \text{there is only one solution, } u_p, \text{ with } u'(0) = 0,\ u'(\pi) = 0\}.$$

THEOREM 2.10. *For any* $\varepsilon > 0$, *if* $\lambda = \lambda_0$, *then there is a solution with* $u'(0) = u'(\pi) = 0$ *and with* $u < \underline{U}(\pi)$ *on* $[0, \pi]$. *Also*, $\lambda_b < \lambda_0$.

Proof. Suppose that $\varepsilon > 0$ and let $\lambda = \lambda_0$. It will be helpful to consider the graph of the set of solutions of $u'' = 0$, shown in Fig. 1b. Starting from $\alpha = -\sqrt{\frac{\lambda}{3}}$ we consider the solution u_α as α is lowered. Let $t_2(\alpha)$ denote the first intersection of this solution with $u = -\sqrt{\frac{\lambda}{3}}$. Then near $\alpha = -\sqrt{\frac{\lambda}{3}}$, $t_2(\alpha)$ is defined and continuous. Further, $u_\alpha' > 0$ on $(0, t_2(\alpha)]$ as long as $t_2(\alpha) < \pi$. The solution cannot be tangent to the line $u = -\sqrt{\frac{\lambda}{3}}$ until we reach a value of where $t_2(\alpha) = \pi$. Let $\hat{\alpha} = \inf\{\alpha \mid t_2(\alpha) = \pi\}$. Suppose that $u_{\hat{\alpha}}'(\pi) = 0$. Since $\lambda = \lambda_0$, it follows that $u_{\hat{\alpha}}''(\pi) = 0$ and we easily calculate that $u_{\hat{\alpha}}'''(\pi) = 0$ while $u_{\hat{\alpha}}''''(\pi) > 0$. But this implies that $u_{\hat{\alpha}} = -\sqrt{\frac{\lambda}{3}}$ at some earlier point, a contradiction. Therefore, $u_{\hat{\alpha}}'(\pi) > 0$. Since $u_\alpha'(\pi) < 0$ for $\alpha < -b$, there is an $\alpha_1 \in (-b, \hat{\alpha})$ with $u'(\pi) = 0$. Then u_{α_1} satisfies the conditions for the solution sought in the theorem.

Further, for $\lambda = \lambda_0$ and sufficiently small $\delta > 0$, $u_{\hat{\alpha}-\delta}(\pi) < \underline{U}(\pi)$ and $u'_{\hat{\alpha}-\delta}(\pi) > 0$. These inequalities then hold for λ slightly less than λ_0, and in turn they imply the continued existence of a periodic solution which remains in the region $u < -\sqrt{\frac{\lambda}{3}}$. Hence, $\lambda_b < \lambda_0$. This proves the theorem, but we note also that a similar argument, or just the use of symmetry, shows that there is a third periodic solution, with $\alpha > \sqrt{\frac{\lambda}{3}}$. ∎

3. RESULTS FOR "LARGE" ε

3.1. A Simple Proof of Chaos

In this section, we show how the shooting method can give a global result about chaos for this equation. Our aim is for a concise statement implying the existence of uncountably many bounded solutions and a natural map between a family of symbol sequences and a set of bounded solutions, where moreover, we can get specific estimates on ranges of λ and ε which support this chaotic behavior. This will mean that we do not prove uniqueness of the solution corresponding to a particular sequence, and hence we can only prove a weaker sensitivity to initial conditions than one obtains from some asymptotic methods. In a later section, in which ε is taken as sufficiently small, more precise results will be obtained.

Our results in the next few sections have one main hypothesis, which we state now.

Condition 3.1. There is an $\bar{\alpha} \in \left(\alpha_0, -\sqrt{\frac{\lambda}{3}}\right)$ such that $u_{\bar{\alpha}}(t)$ increases monotonically as t increases from 0, and crosses $u = b$ before $t = \frac{\pi}{2}$.

Recall that b is positive and satisfies $b^3 - \lambda b - 1 > 0$. An easy phase plane argument (given in Section 3.4) shows that if $\lambda \geqslant \lambda_0$, then Condition 3.1 is satisfied for sufficiently small ε. Later we will show, with a bit more work, that Condition 3.1 is satisfied in a specific range of ε and λ.

We use $\bar{\alpha}$ as given in Condition 3.1 to define a family $\{\hat{w}_k\}$ of "special" solutions as follows:

$$\hat{w}_k(t) = (-1)^k u_{\bar{\alpha}}(t - k\pi). \tag{3.1}$$

Thus, each $\hat{w}_k$ is a translation, and for odd k a reflection, of the solution $u_{\bar{\alpha}}$. It is easily seen that $(-1)^k \hat{w}_k$ has its global minimum at $k\pi$ and on each side of this point increases monotonically to $u = b$ before $|t - k\pi| = \frac{\pi}{2}$.

Let w_k denote the restriction of $\hat{w}_k$ to the interval $[s_k, S_k]$ in which $|\hat{w}_k| \leqslant b$. In Fig. 2 we plot several of the functions w_k.

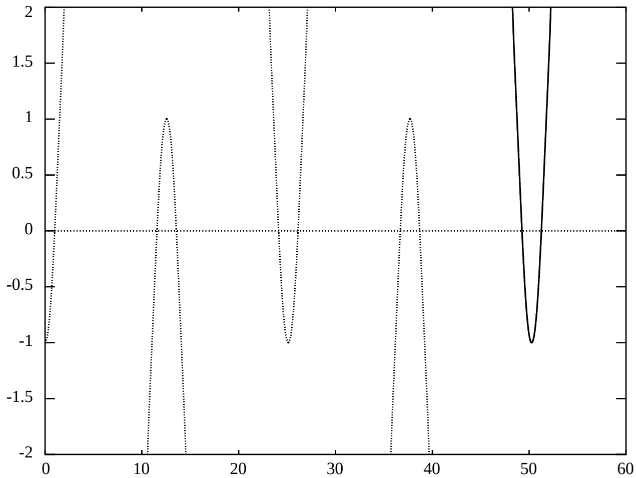

FIG. 2. Graphs of w_0, w_1, w_2, w_3, and w_4.

We can now state our result.

THEOREM 3.2. *Assume that $\lambda > \lambda_0$ and that Condition 3.1 holds. Let $\Sigma = \{\sigma_k\}_{k=1..\infty}$ be any sequence of positive integers such that $\sigma_{k+1} - \sigma_k \geqslant 2$. Then there is an $\alpha_\Sigma \in (\alpha_0, \bar{\alpha})$ such that u_{α_Σ} exists on $(-\infty, \infty)$, and on $(0, \infty)$ the graph of u_{α_Σ} intersects the graph of each w_{σ_k}, but does not intersect the graph of any w_j with $j \notin \Sigma$.*

Remark. 3.3 We allow Σ to be finite, or even empty.

Proof. We define two functions, f_+ and f_-, as follows:

$$f_+(t) = \begin{cases} w_k(t) & \text{if } k \text{ is even and } s_k \leqslant t \leqslant S_k, \\ & b \text{ otherwise,} \end{cases}$$

$$f_-(t) = \begin{cases} w_k(t) & \text{if } k \text{ is odd and } s_k \leqslant t \leqslant S_k, \\ & -b \text{ otherwise.} \end{cases}$$

A key observation is that no solution can intersect the curve f_- tangentially from above, or the curve f_+ tangentially from below. This is because $u'' < 0$ when $u = -b$, $u'' > 0$ when $u = b$, and no two distinct solution graphs can be tangent to each other. Also, no solution can be tangent to either of the lines $u = \pm b$ from between these two lines.

Further, we will need two other functions, g_+ and g_-, where $g_- < 0 < g_+$. These are defined by

$$g_+(t) = \max\left\{\sqrt{\frac{\lambda}{3}},\ f_-(t)\right\}, \qquad g_-(t) = \min\left\{-\sqrt{\frac{\lambda}{3}},\ f_+(t)\right\}.$$

As in Fig. 3, the function f_+ can be described as the function whose graph is the line $u = b$ except at downward bumps when this line meets a w_k with k even. Also, the graph of g_+ is the line $u = \sqrt{\frac{\lambda}{3}}$ except for upward bumps when this line meets the graph of w_k for some odd k. The functions f_- and g_- are reflections and translations of f_+ and g_+. Recall that in Condition 3.1 we required that $|\bar{\alpha}| > \sqrt{\frac{\lambda}{3}}$. No solution can be tangent to g_+ from above or to g_- from below. Also, no solution can be tangent to $u = \sqrt{\frac{\lambda}{3}}$ from above or to $u = -\sqrt{\frac{\lambda}{3}}$ from below.

Assuming that the sequence Σ is chosen, we shall define a sequence of closed intervals $\{I_k\}$ with the following properties (see Fig. 4):

(i) For each k, $\alpha \in I_k$ implies that u_α intersects $w_{\sigma_1}, \ldots, w_{\sigma_k}$ and no other w_j with $j < \sigma_k$.

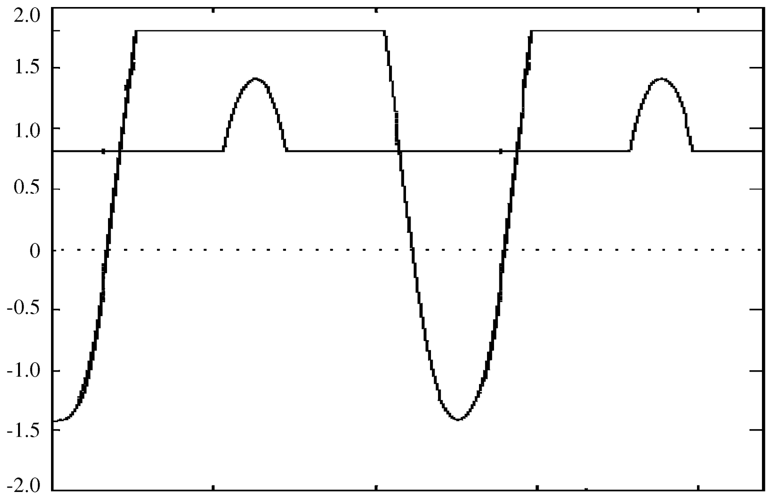

FIG. 3. Graphs of f_+ and g_+.

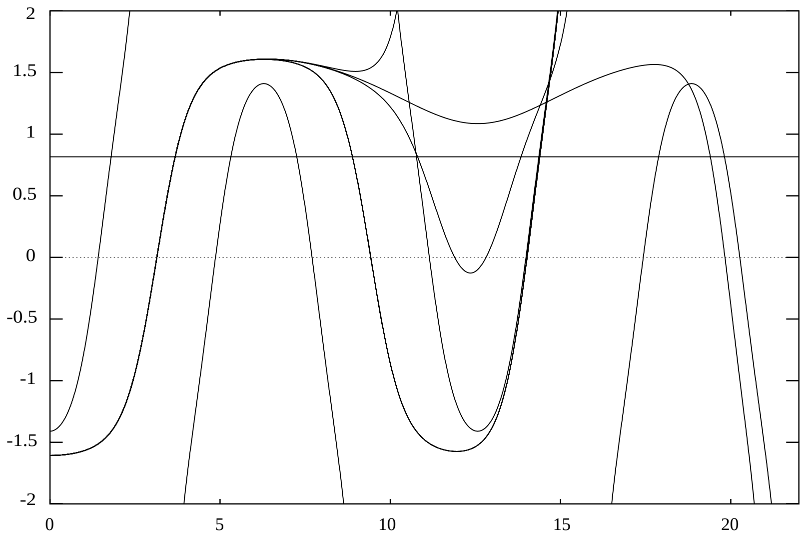

FIG. 4. Several solutions used in the shooting process, with the scale $t \to \frac{t}{\varepsilon}$, $\varepsilon = 0.5$. Solutions shown are $u_{\alpha_1}, u_{\beta_1}$, u_{μ_1}, and u_{θ_1}, together with w_0, w_1, w_2, and w_3.

(ii) Let $I_k = [\beta_k, \alpha_k]$, and for $\alpha \in I_k$ let $t_k(\alpha)$ denote the first t such that $u_\alpha(t) = w_{\sigma_k}(t)$. Then either $t_k(\alpha_k) = s_{\sigma_k}$ and $t_k(\beta_k) = S_{\sigma_k}$, or $t_k(\beta_k) = s_{\sigma_k}$ and $t_k(\alpha_k) = S_{\sigma_k}$.

(iii) $I_k \subset I_{k-1}$.

Since the intervals will be closed, their intersection will be nonempty, and a point $\alpha_\Sigma \in \bigcap_{k=1}^{\infty} I_k$ will have the properties stated in the theorem. We do not prove that the intersection is only one point. We will be able to do this for sufficiently small ε.

We will choose I_1 as a subinterval of $(-b, \bar{\alpha})$. Suppose, first, that σ_1 is even. For α close to $\bar{\alpha}$, u_α intersects f_+ before $t = \frac{\pi}{2}$, and therefore before u_α could intersect w_1. Let $t_1(\alpha)$ denote the first intersection of u_α with f_+. Then $t_1(\cdot)$ is continuous in some maximal subinterval $(\beta, \bar{\alpha}]$ of $(-b, \bar{\alpha}]$. However, $\beta > -b$, for solutions u_α with α close to $-b$ do not intersect f_+ at all. (Instead they decrease monotonically to below $u = -b$ and then to $-\infty$.) The nontangency of u_α with f_+ implies that

$$\limsup_{\alpha \to \beta^+} t_1(\beta) = \infty.$$

We do not know if $t_1(\alpha)$ is monotone increasing. However, there must be an interval $I_1 = [\beta_1, \alpha_1]$ such that $t_1(\alpha_1) = s_{\sigma_1}$, $t_1(\beta_1) = S_{\sigma_1}$, and $s_{\sigma_1} < t_1(\alpha)$

$< S_{\sigma_1}$ in (β_1, α_1), $t_1(\alpha) < S_{\sigma_1}$ on $(\beta_1, \bar{\alpha}]$. As long as $t_1(\alpha)$ is continuous, it must remain the first intersection of u_α with f_+, because of the nontangency of u_α with f_+. Further, while $t_1(\alpha)$ is continuous the solution u_α cannot intersect f_- in (0, $t_1(\alpha)$), because of nontangency with f_-. A similar argument is used to construct I_1 if σ_1 is odd. In this case, we have $t_1(\beta_1) = s_{\sigma_1}$ and $t_1(\alpha_1) = S_{\sigma_1}$.

We now assume that a (decreasing nested) sequence of closed intervals $I_1, \dots, I_n$ has been constructed with properties (i)–(iii) for $k = 1, \dots, n$. We wish to construct $I_{n+1} \subset I_n$. We will consider the case where σ_n is even and σ_{n+1} is odd. We will also assume that $t_n(\alpha_n) = s_{\sigma_n}$ while $t_n(\beta_n) = S_{\sigma_n}$. We will show that we can construct $[\alpha_{n+1}, \beta_{n+1}]$ so that $t_{n+1}(\alpha_{n+1}) = S_{\sigma_{n+1}}$ and $t_{n+1}(\beta_{n+1}) = s_{\sigma_{n+1}}$, or we could make a different choice which would result in $t_{n+1}(\alpha_{n+1}) = s_{\sigma_{n+1}}$, $t_{n+1}(\beta_{n+1}) = s_{\sigma_{n+1}}$. (Hence, for any given sequence Σ there will be many solutions with the properties (i)–(iii). This does not by itself imply a reduced sensitivity to initial conditions, because these solutions may be separated from each other. This is discussed in a separate section below.)

The assumption that σ_n is even means that the function w_{σ_n} is a downward-pointing spike and $w_{\sigma_n}(s_{\sigma_n}) = w_{\sigma_n}(S_{\sigma_n}) = b$, while taking σ_{n+1} odd means that $w_{\sigma_{n+1}}$ is an upward-pointing spike.

To define I_{n+1}, in the case where σ_n is even, let $\mu_n = \sup\{\alpha < \alpha_n \mid u_\alpha(t_n(\alpha)) = \sqrt{\frac{\lambda}{3}}\}$. Then $t_n(\alpha) \in (s_{\sigma_n}, \sigma_n\pi)$, $u_\alpha(t_n(\alpha)) \in (\sqrt{\frac{\lambda}{3}}, b)$ for $\mu_n < \alpha < \alpha_n$, and $u'_{\mu_n}(t_n(\mu_n)) < 0$. The last inequality is true because u_{μ_n} does not intersect $w_{\sigma_n - 1}$, an upward pointing spike, and $|\bar{\alpha}| > \sqrt{\frac{\lambda}{3}}$. (If we wished to have $t_{n+1}(\alpha_{n+1}) = s_{\sigma_{n+1}}$ we would set $\mu_n = \inf\{\beta > \beta_n \mid u_\alpha(t_n(\alpha)) = \sqrt{\frac{\lambda}{3}}\}$. In this case $t_n(\alpha) \in (\sigma_n\pi, S_n)$, $u_\alpha(t_n(\alpha)) \in (\sqrt{\frac{\lambda}{3}}, b)$ for $\beta_n < \alpha < \mu_n$, and $u'_{\mu_n}(t_n, \mu_n) > 0$. Slight changes are necessary if σ_n is odd.)

Consider values of α near to μ_n. Since $u_{\mu_n}(t_n(\mu_n)) = \sqrt{\frac{\lambda}{3}}$, $s_n < t_n(\mu_n) < \sigma_n\pi$, and $u'_{\mu_n}(t_n(\mu_n)) < 0$, we can define $\tau_n(\alpha)$ continuously by the equations $u_\alpha(\tau_n(\alpha)) = g_+(\tau_n(\alpha))$, $\tau_n(\mu_n) = t_n(\mu_n)$. Then $\tau_n(\cdot)$ will be continuous in some maximal interval around μ_n. However, as α increases from μ_n, $\tau_n(\alpha)$ must tend to infinity, since it is not defined at $\alpha = \alpha_n$, where the solution u reaches b at $s_n = t_n(\alpha_n)$ and never decreases below b after that.

Let

$$\theta_n = \sup\{\alpha > \mu_n \mid \tau_n \text{ is continuous on } [\mu_n, \alpha] \text{ and } \tau_n(\alpha) = (\sigma_n + 1)\pi\}.$$

The interval I_{n+1} will be a subinterval of $[\theta_n, \alpha_n]$.

Observe that u_{θ_n} crosses w_{σ_n+1}, from above, at its maximum point, at $t = (\sigma_n + 1)\pi$. This is also an intersection of u_{θ_n} with f_-. Let the intersection of this solution with f_- be denoted by $\rho = \rho_n(\theta_n)$, and extend ρ_n as a function of α continuously for $\alpha \geqslant \theta_n$, ρ_n being defined as the intersection of u_α with f_-, as long as it remains continuous. The solution u_α may possibly intersect f_- at earlier points, but ρ_n is defined uniquely by requiring that $\rho_n(\theta_n) = (\sigma_n + 1)\pi$.

We see that as α increases from θ_n the function ρ_n must eventually be undefined, since it is not defined at α_n. So, it must increase (not necessarily monotonically), and there must be some closed subinterval of (θ_n, α_n) in which $\rho_n(\alpha)$ lies in the interval $[s_{\sigma_{n+1}}, S_{\sigma_{n+1}}]$ and moves from the left end of this interval to the right end (not necessarily monotonically) as α increases. This subinterval of (θ_n, α_n) is chosen for the interval $I_{n+1} = [\beta_{n+1}, \alpha_{n+1}]$. We define it unambiguously by choosing the subinterval with the given properties which lies nearest to θ_n. The possibility of an infinite set of subintervals with these properties is precluded by bounds on the variables for a given ε. The construction of I_{n+1} shows that it satisfies conditions (i)–(iii).

A similar construction will give I_{n+1} in the case where σ_n and σ_{n+1} are both even. In this case, we can obtain $t_{n+1}(\alpha_{n+1}) = s_{\sigma_{n+1}}$ by decreasing α from α_n and observing that the intersection of u_α with $u = f_+$ must tend to ∞. before we reach θ_n. The case where σ_n is odd is also handled similarly. At each step we can obtain $t_n(\alpha_n) = s_{\sigma_n}$ if σ_n is even and $t_n(\alpha_n) = S_{\sigma_n}$ if σ_n is odd. This completes the induction step and the proof of Theorem 3.2. ∎

3.2. "*Kneading*" *Theory*

A one-parameter shooting process as in the proof of Theorem 3.2 inherently gives an ordering of the initial conditions corresponding to different sequences. This sort of ordering is related to "kneading theory" [GH]. In earlier work on similar problems formal results about this were stated, for symbol sequences on two symbols [HM1, HM2, HT1, HT2]. Here, we will describe the theory only for the solutions found in Theorem 3.2. Further solutions found below would make the description more complicated and so this part of the theory will not be discussed in those cases.

Notice that in the induction procedure used in the proof of Theorem 3.2, repeated use was made of lowering or raising α and letting the intersections with the sets f_+ or f_- tend to infinity. There was never an assumption that these intersections moved monotonically. However, the intersection will pass through some particular w_k before it first intersects w_{k+2}, the next spike pointing in the same direction as w_k. The point of intersection might retreat along the t-axis and intersect w_k again. This would increase the number of solutions corresponding to some particular sequence, but does not prevent our giving an order to those chosen as in the proof of Theorem 3.2. Our repeated choice to have $t_n(\alpha_n) = s_{\sigma_n}$, rather than S_{σ_n}, makes the relation between solutions and sequences easier to describe. (If $\sigma_1 = 1$ then we could not make this choice at step 1, but could thereafter.) The ordering of solutions corresponding to sequences by the algorithm used in the proof is well defined because of the definitions of μ_n, θ_n, etc. This is despite the fact that we do not know that the nested intervals converge to single points. Any choice of points within the limit intervals will give the same ordering. We have the following corollary of the proof of Theorem 3.2:

THEOREM 3.4. *Suppose that Σ_1 and Σ_2 are two sequences of positive integers as in the statement of Theorem* 3.2. *Suppose that for some $k \geqslant 0$, $\sigma_i(\Sigma_1) = \sigma_i(\Sigma_2)$ if $1 \leqslant i \leqslant k$. If $\sigma_{k+1}(\Sigma_1)$ is even and $\sigma_{k+1}(\Sigma_2)$ is odd, then $\alpha(\Sigma_1) > \alpha(\Sigma_2)$. If $\sigma_{k+1}(\Sigma_1) > \sigma_{k+1}(\Sigma_2)$ and both of these integers are even, then $\alpha(\Sigma_1) < \alpha(\Sigma_2)$. If $\sigma_{k+1}(\Sigma_1) > \sigma_{k+1}(\Sigma_2)$ and both integers are odd, then $\alpha(\Sigma_1) > \alpha(\Sigma_2)$.*

3.3. Further Periodic Solutions, and Symbolic Dynamics with Five Symbols

Earlier we showed that for $\lambda > \lambda_0$, there are at least three solutions with period 2π. Numerically, however, we find five periodic solutions $u_1, \ldots, u_5$, with $u_1 < u_2 < u_3 < u_4 < u_5$. Figure 5 is a graph of these solutions when $\varepsilon = 1$, $\lambda = 2$.

The solution u_3 is the solution u_{p} which is antisymmetric around $\frac{\pi}{2}$, found in Theorem 2.6. The solutions u_1 and u_5 are the ones found in Proposition 2.4. These are the same ones found in [AMPP], at least for small ε, and

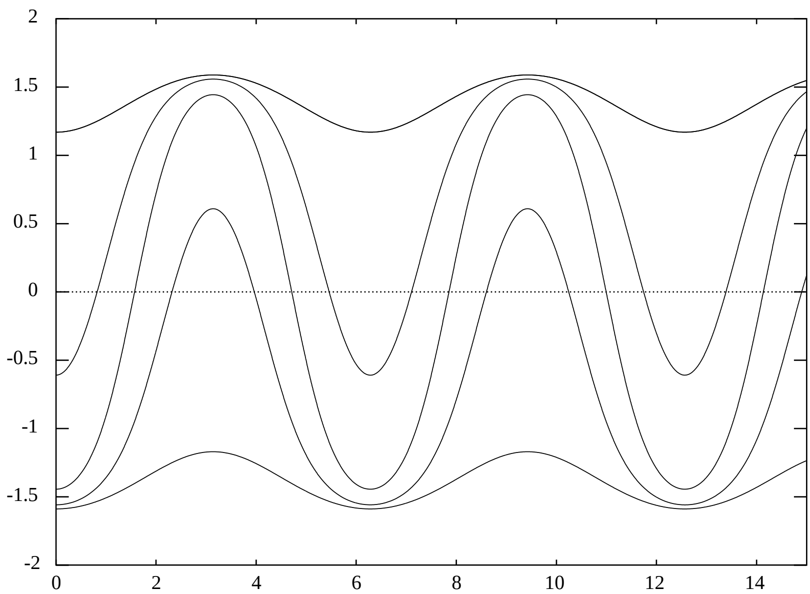

FIG. 5. Periodic solutions, $u_1, u_2, u_3 (= u_{\mathrm{p}}), u_4$; and u_5. Here $\varepsilon = 1$, $\lambda = 2$.

correspond to stable steady states for (1.2)–(1.3). The solutions u_2 and u_4 seem to be new (although their existence is strongly suggested by the existence of three stable steady states). Other solutions are found which "follow" one or another of these solutions over long intervals, switching in almost arbitrary ways from one to another. To state a precise theorem, we have to capture what it means to "follow" a solution over an interval in a way that sharply distinguishes different patterns. We need to identify disjoint open sets of solutions which, over a finite interval, follow the solutions in different orders. We are able to do this because of the various nontangency conditions stated above.

The following result is stated without reference to $u_1, \dots, u_5$, or even knowledge that u_2 and u_4 exist. It will be seen later that existence of u_2 and u_4 is an easy consequence of the technique used to prove the result.

THEOREM 3.5. *Suppose that the hypotheses of Theorem* 3.2 *are satisfied.* (*That is,* $\lambda > \lambda_0$ *and Condition* 3.1 *holds.*) *Let* $\{w_k\}$ *be the family of* "*spike*" *solutions defined by Eq.* (3.1). *Let* $\Omega = \{\omega_k\}$ *be a sequence chosen from the integers* $\{1,2,3,4,5\}$. *Assume that no* 1 *or* 2 *is followed immediately by a* 4 *or* 5, *and no* 4 *or* 5 *is followed immediately by a* 1 *or* 2. (*There must be an intervening* 3.) *Then there is a solution* $u = u_\Omega$ *with the following properties*:

(i) *If* $\omega_k = 1$ *or* 2 *then* u *intersects* w_{2k-1} *at two points and does not intersect* w_{2k}.

(ii) *If* $\omega_k = 1$ *then* $u < g_-$ *in* $((2k-2)\pi, 2k\pi)$ *while if* $\omega_k = 2$, *then* $u > g_-$ *somewhere in this interval.*

(iii) *If* $\omega_k = 3$ *then* u *does not intersect either* w_{2k-1} *or* w_{2k}.

(iv) *If* $\omega_k = 4$ *or* 5, *then* u *does not intersect* w_{2k-1} *but does intersect* w_{2k}, *at two points.*

(v) *If* $\omega_k = 4$ *then* u *falls below* g_+ *in* $((2k-1)\pi, (2k+1)\pi)$, *while if* $\omega_k = 5$, *then* $u > g_+$ *in this interval.*

Proof. The proof is a refinement of the proof of Theorem 3.2. The notation is different, however, for in the statement of Theorem 3.5 all of the w_k are included, not just those picked out by some index set $\{\sigma_k\}$. As before, $s_k < S_k$ will be the endpoints of the graphs of w_k. The functions f_- and f_+, g_- and g_+, are the same as before. Also, $t_k(\alpha)$ will denote the first intersection of u_α with w_k.

Rather than carry out a formal induction, we will show how to construct a solution corresponding to a sequence beginning with $\omega_1 = 4$ or 5, $\omega_2 = 3$, $\omega_3 = 1$ or 2. Hence we want a solution which misses w_1, intersects w_2, misses w_3 and w_4, intersects w_5, and does not intersect w_6. It will then be apparent how to deal with the general case. In the notation of Theorem 3.2, we are starting with $\sigma_1 = 2,\ \sigma_2 = 5$.

If $\omega_1 = 5$, then we proceed in the same way as in the proof of Theorem 3.2. The induction step there (going from $n = 1$ to 2) produced an interval I_2 in which the solutions u_α do not cross g_+ in $[\pi, 3\pi]$. Further, the solutions in this interval will not intersect w_3 or w_4 and will intersect the two endpoints of w_5 when α is at the endpoints of I_2. The refinement of this interval according to whether $\omega_3 = 1$ or 2 will be similar to the way we handle the case $w_1 = 4$, so we turn to that case now.

When $\omega_1 = 4$, we follow the proof of Theorem 3.2 up to the definition of μ_1. However now we set

$$\mu_1 = \inf\left\{\alpha > \beta_1 \mid t_1(\alpha) < 2\pi \text{ and } u_\alpha(t_1(\alpha)) = \sqrt{\frac{\lambda}{3}}\right\}.$$

Then for $\alpha \in [\beta_1, \mu_1]$, u_α intersects w_2 to the right of the first point where $w_2 = \sqrt{\frac{\lambda}{3}}$. We now let $\tau_1(\alpha)$ be the intersection of u_α with g_+ defined by continuity from the point $u_{\mu_1}(t_1(\mu_1))$. Set

$$\theta_1 = \inf\{\alpha > \mu_1 \mid \tau_1(\alpha) = 3\pi\}$$

and

$$\phi_1 = \sup\{\alpha < \mu_1 \mid t_1(\alpha) = 2\pi\}.$$

These are defined because u_{α_1} does not intersect g_+ after π, and because $t_1(\alpha)$ varies between $\tau_1(\mu_1)$ and S_1 as α moves from μ_1 to β_1. Then, $\phi_1 \in (\beta_1, \theta_1)$ and for $\alpha \in [\phi_1, \theta_1)$ u_α intersects w_2 and also intersects g_+ in the interval $(\pi, 3\pi)$.

The solution u_{ϕ_1} intersects w_2 at its minimum point, at $t = 2\pi$, and from there increases to cross b before $\frac{5\pi}{2}$. (This is shown by a comparison with w_2, as in the proof of Lemma 2.7.) The solution u_{θ_1} intersects w_2 twice, intersects w_3 at its maximum point and then decreases to cross $-b$ before $\frac{7\pi}{2}$. Following the same procedure as in the proof of Theorem 3.2, we lower α from θ_1 and find an interval $I_2 = [\beta_2, \alpha_2] \subset (\phi_1, \theta_1)$ such that $u_{\alpha_2}(s_3) = -b$, $u_{\beta_2}(S_3) = -b$, and for $\alpha \in (\beta_2, \alpha_2)$, u_α intersects g_+ in $(\pi, 3\pi)$ and also intersects w_3. Other cases, and succeeding steps, being similar, this completes the proof of Theorem 3.5. ∎

We can also use this process to prove the existence of u_2 and u_4:

THEOREM 3.6 *If* $\lambda > \lambda_0$ *and Condition* 3.1 *holds, then there are at least five* 2π*-periodic solutions of* 1.4, $u_1, u_2, u_3 (= u_p), u_4$ *and* u_5*, all with* $u'(\pi) = 0$. *Further, these solutions are ordered by* $u_1 < u_2 < u_3 < u_4 < u_5$.

Proof. We already know that u_1, u_3, and u_5, which exist by Proposition 2.4, are increasing in $(0, \pi)$. We give two proofs of the existence of u_2 and u_4. The first is a little shorter, while the second shows in addition that u_2 and u_4 are increasing on $(0, \pi)$.

Proof 1. We will find a solution $u = u_\alpha$ with the property that $\alpha < \alpha_p$, $u'(\pi) = 0$, $u(\pi) > g_-$, and u intersects w_1. (None of the three solutions u_1, u_3, and u_5 have all of these properties.) We assume in Theorem 3.5 that $\omega_1 = 2$. Then in the construction used in the proof of Theorem 3.5 we find μ_1 such that $u' > 0$ on $(0, t_1(\mu_1)]$ and $u_{\mu_1}(t_1(\mu_1)) = g_-$. We also find ϕ_1 such that $t_1(\phi_1) = \pi$, and necessarily, $u'_{\phi_1}(t_1(\phi_1)) < 0$. If we decrease α from ϕ_1 we come to a first (largest) α, say $\hat{\alpha}$, where $u_{\hat{\alpha}}$ first intersects g_- at $t = \pi$. At that point, $u'_{\hat{\alpha}}(\pi) > 0$. Somewhere between $\hat{\alpha}$ and ϕ_1 there must be an α with $u'_\alpha(\pi) = 0$, $u_\alpha(\pi) > g_-$. This solution must also intersect w_1, and this gives the desired fourth periodic solution (beyond those constructed in Proposition 2.4.) The fifth is obtained by translation and reflection. From Corollary 2.9, it follows that $u_1 < u_2 < u_3 < u_4 < u_5$.

Proof. If $\alpha = u_1(0)$, then $v(t) = \frac{\partial u_\alpha}{\partial \alpha}|(t) > 0$ and $v'(t) > 0$ for $t \in (0, \pi]$, and $u_1' > 0$ in $(0, \pi)$. Also, u_1 intersects w_1 in $(0, \pi)$, and it follows that for $\alpha > u_1(0)$ and sufficiently close to $u_1(0)$, u'_α is also positive on $(0, \pi)$, and again u_α intersects w_1 at some point in $(0, \pi)$. Let

$$\alpha_2 = \sup\{\hat{\alpha} \in (u_1(0), u_p(0)) \mid \text{for } u_1(0) < \alpha \leqslant \hat{\alpha}, \quad u'_\alpha > 0 \text{ on } (0, \pi) \\ \text{and } u_\alpha \text{ intersects } w_1 \text{ at some point in } (0, \pi)\}.$$

Then $\alpha_2 < u_p(0)$ since u_p does not intersect w_1 at all.

By continuity, and since no u_α is tangent to w_1, $u_2 := u_{\alpha_2}$ intersects w_1 in $(0, \pi]$ and $u_2' \geqslant 0$ in $[0, \pi]$. The first intersection of u_2 with w_1 is either at π or in $(0, \pi)$. In the first case $u_2'(\pi) > 0$, which implies $(u_2 - w_1)'(\pi) > 0$ and so $u_2 < w_1$ just before π, a contradiction. Hence u_2 first intersects w_1 in $(0, \pi)$. Then by the definition of α_2, it follows that $u_2' = 0$ at some first $\tilde{t} \in (0, \pi]$. We claim that $\tilde{t} = \pi$. If $0 < \tilde{t} < \pi$, then $u_2''(\tilde{t}) = 0$ for otherwise, $u_2''(\tilde{t}) < 0$ and this implies that $u_2' < 0$ just after $\tilde{t}$, a contradiction. So $u_2'''(\tilde{t}) = -\sin \tilde{t} < 0$ which gives $u_2' < 0$ just before and after $\tilde{t}$, again a contradiction. This shows that $u_2'(\pi) = 0$ so that u_2 is a periodic solution which satisfies $u_1(0) < u_2(0) < u_p(0)$ and $u_2' > 0$ in $(0, \pi)$. Further, u_2 intersects w_1 in $(0, \pi)$, which implies that $u_2(\pi) < w_1(\pi)$. Let $u_4(t) = -u_2(t + \pi)$. From Theorem 2.6 and Corollary 2.9, it again follows that $u_1 < u_2 < u_3 < u_4 < u_5$. ∎

3.4. Verification of Condition 3.1 for a Range of Parameters

It is trivial to verify Condition 3.1 for $\lambda > \lambda_0$ and sufficiently small ε. This is most easily seen by making the change of variables $\tau = \frac{t}{\varepsilon}$, letting $v(\tau) = u(t)$, and considering the resulting equation

$$\ddot{v} = v^3 - \lambda v + \cos \varepsilon\tau. \tag{3.2}$$

Then consider the phase plane for the limiting equation

$$\ddot{v} = v^3 - \lambda v + 1.$$

There are three equilibria in the $v, \dot{v}$ plane, $p_i = (\bar{\alpha}_i, 0)$, $i = 1, 2, 3$, where $\bar{\alpha}_1 < 0 < \bar{\alpha}_2 < \bar{\alpha}_3$. The outer two, p_1 and p_3 are saddle points while p_2 is a center. There is a homoclinic orbit at p_3, so that the left branch of the unstable manifold at p_3 is bounded, with v taking its minimum value at a point $(\alpha^*, 0)$ where $\bar{\alpha}_1 < \alpha^* < \bar{\alpha}_2$, while the unstable manifold at p_1 is unbounded to the right. Suppose that $v(0) = \bar{\alpha} \in (\bar{\alpha}_1, \alpha^*)$. Then $\dot{v} > 0$ for $t > 0$ as long as the solution is defined, and $v \to \infty$ in finite time. Continuity with respect to ε implies that for small enough ε, the solution of (3.2) with $v(0) = \bar{\alpha}$, $\dot{v}(0) = 0$ will increase monotonically and cross $v = b$ before $\tau = \frac{\pi}{2\varepsilon}$. This verifies Condition 3.1 for small ε.

Next we give an estimate on ε given $\lambda > \lambda_0$ to ensure that Condition 3.1 holds. This estimate of ε is by no means best.

LEMMA 3.7. *For any* $\lambda \geqslant \lambda_0$ *let* $b = b(\lambda) = \sqrt{\lambda + \frac{1}{2\lambda}}$ *and* $\varepsilon_\lambda = \frac{\pi}{3T_\lambda}$, *where* $T_\lambda = 2\sqrt{b + \sqrt{\lambda}}$ *for* $\lambda_0 \leqslant \lambda < 4$ *and*

$$T_\lambda = 2\sqrt{2} + \frac{\sqrt{2}\ln(\sqrt{\lambda} - 1)}{\sqrt{\lambda}} + 2(\sqrt{b + \sqrt{\lambda}} - \sqrt{2\sqrt{\lambda} - 2}) \tag{3.3}$$

for $\lambda \geqslant 4$. *Let* v *be the solution of* (3.2) *with* $v(0) = -\sqrt{\lambda}$ *and* $\dot{v}(0) = 0$. *If* $0 < \varepsilon \leqslant \varepsilon_\lambda$ *in* (3.2), *then there is a* $T \in (0, T_\lambda)$ *such that*

$$\dot{v} > 0 \qquad \textit{in } (0, T] \subset \left(0, \frac{\pi}{3\varepsilon}\right], \qquad \textit{and} \qquad v(T) = b. \tag{3.4}$$

Proof. Let

$$T = \sup\left\{\tau \in \left(0, \frac{\pi}{3\varepsilon}\right) \middle| \dot{v} > 0,\ v < b \text{ in } (0, \tau]\right\}.$$

Since $\ddot{v}(0) = 1$, it follows that T is well defined. Then for $\tau \in (0, T)$, $\cos(\varepsilon\tau) \geqslant 1/2$ and then (3.2) gives $\ddot{v} \geqslant v^3 - \lambda v + 1/2$. Multiply this inequality by $2\dot{v}$, then integrate over $[0, \tau]$ and use $v(0) = -\sqrt{\lambda}$, $\dot{v}(0) = 0$ to give

$$(\dot{v})^2 \geqslant \tfrac{1}{2}v^4 - \lambda v^2 + v - \left(\tfrac{1}{2}\lambda^2 - \lambda^2 - \sqrt{\lambda}\right) = \tfrac{1}{2}(v^2 - \lambda)^2 + (v + \sqrt{\lambda}) \tag{3.5}$$

and hence, for $\tau \in (0, T]$,

$$\dot{v} \geqslant \sqrt{\tfrac{1}{2}(v^2 - \lambda)^2 + (v + \sqrt{\lambda})}. \tag{3.6}$$

This implies that

$$T < \int_{-\sqrt{\lambda}}^{b} \frac{dv}{\sqrt{\frac{1}{2}(v^2 - \lambda)^2 + (v + \sqrt{\lambda})}} \tag{3.7}$$

and $\dot{v}(T) > 0$.

If $\lambda_0 \leqslant \lambda < 4$, then $T < \int_{-\sqrt{\lambda}}^{b} \frac{1}{\sqrt{v+\sqrt{\lambda}}}\, dv = 2\sqrt{b + \sqrt{\lambda}} = T_\lambda$, which proves (3.4) for $0 < \varepsilon < \varepsilon_\lambda$. Assume that $\lambda \geqslant 4$. Note that

$$\int_{-\sqrt{\lambda}}^{b} \frac{dv}{\sqrt{\frac{1}{2}(v^2 - \lambda)^2 + (v + \sqrt{\lambda})}} \leqslant I_1 + I_2 + I_3,$$

where

$$I_1 := \int_{-\sqrt{\lambda}}^{-\sqrt{\lambda}+2} \frac{dv}{\sqrt{v + \sqrt{\lambda}}} = 2\sqrt{2},$$

$$I_2 := \int_{-\sqrt{\lambda}+2}^{\sqrt{\lambda}-2} \frac{\sqrt{2}\, dv}{\lambda - v^2} = \frac{\sqrt{2}\ln(\sqrt{\lambda} - 1)}{\sqrt{\lambda}},$$

and

$$I_3 := \int_{\sqrt{\lambda}-2}^{b} \frac{dv}{\sqrt{v + \sqrt{\lambda}}} = 2\left(\sqrt{b + \sqrt{\lambda}} - \sqrt{2\sqrt{\lambda} - 2}\right).$$

It follows from (3.3) that $T < T_\lambda$ and therefore (3.4) follows from the definition of T and the assumption that $0 < \varepsilon \leqslant \varepsilon_\lambda$. ∎

Remark. 3.8. Since $\lim_{\lambda \to \infty} T_\lambda = 2\sqrt{2}$, it follows from Lemma 3.7 that for sufficiently large $\lambda > 0$, Condition 3.1 holds for $0 < \varepsilon < \frac{\pi}{6\sqrt{2}}$. Also, easy numerical estimates (assisted by a computer algebra program!) show that for any $\lambda \geqslant \lambda_0$, Condition 3.1 holds if $0 < \varepsilon < \frac{1}{4}$.

4. RESULTS FOR SMALL ε

4.1. Asymptotic Form of Periodic Solutions as $\varepsilon \to 0$

One of the key points made in the work of Angenent, Mallet-Paret, and Peletier is that as $\varepsilon \to 0$, the solution which we have called u_{p}, or u_3, tends to the lower root, $\underline{U}(t)$, of $u'' = 0$ in $[0, \frac{\pi}{2})$ and to the upper root, which we have denoted by $\bar{U}(t)$, in $(\frac{\pi}{2}, \pi]$. (This was for the case $\lambda > \lambda_0$.) We will need this result here. Since one of the aims of this paper is to give elementary proofs of their results, with no reliance on infinite-dimensional analysis or partial differential equations, we give a new proof. The result is obtained from the following lemma, where we make no assumption on λ. For any λ, the equation $u^3 - \lambda u + \cos(t) = 0$ has a smallest solution $u = \underline{U}(t)$ which is continuous in any interval $I_k = \left[\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2}\right]$ with k an even integer. When k is odd, $\bar{U}$ is continuous in I_k.

LEMMA 4.1. *For some even integer k let J_k and M_k be closed intervals with $M_k \subset int(J_k) \subset I_k$ (where int denotes interior). Then for any $\delta > 0$, there is an ε_0 such that if $0 < \varepsilon < \varepsilon_0$, and if u is a solution of (1.4) with $-b \leqslant u \leqslant 0$ in J_k, then $|u(t) - \underline{U}(t)| < \delta$ in M_k. If k is odd, a similar result holds, stating that solutions which are positive and bounded by b in J_k are close to $\bar{U}$ in M_k.*

If we recall that u_{p} is an even function, we see that u_{p} must be close to the lowest root $\underline{U}(t)$ of $u'' = 0$ on any interval of the form $\left[0, (1-\delta)\frac{\pi}{2}\right]$ as $\varepsilon \to 0$. This is true also for any bounded solution with $u(0) < u_{\mathrm{p}}(0)$.

Proof. Consider the case k even. The branch $\underline{U}(t)$ is strictly negative in J_k. Further, given a sufficiently small $\delta > 0$, depending on λ and J_k, the quantity

$$f_{\max} = \max_{\substack{\underline{U}(t) - \delta \geqslant u \\ t \in J_k}} f(t, u)$$

is negative and the quantity

$$f_{\min} = \min_{\substack{\underline{U}(t) + \delta \leqslant u \leqslant 0 \\ t \in J_k}} f(t, u)$$

is positive. It follows ε_0 can be chosen so that if $u(t) \leqslant \underline{U}(t) - \delta$ for some $t \in M_k$ and $u'(t) \leqslant 0$, then u crosses $u = -b$ as t increases in J_k, while if $u'(t) > 0$, then u crosses $-b$ in J_k as t decreases. Similarly, if $u(t) \geqslant \underline{U}(t) + \delta$, then u will cross zero before t leaves J_k, either forward or backward depending on the sign of u'. The case k odd is similar. This proves Lemma 4.1. ∎

The remaining results in this section give more information about the asymptotic behavior of solutions of (1.4) as $\varepsilon \to 0$ when $\lambda > \lambda_0$. They will be needed in the next section.

LEMMA 4.2. *Assume that* $\lambda > \lambda_0$. *Let* $K = \sqrt{\underline{U}(\pi)^2 - \frac{\lambda}{3}}$. *Then*

(i) *There are* $M_0 > 0$ *and* $\varepsilon_0 > 0$ *such that if* $0 < \varepsilon < \varepsilon_0$ *and* u *is a solution of* (1.4) satisfying $u < -\sqrt{\frac{\lambda}{3}}$ *over an interval* (c, d) *with* $-\infty \leqslant c < d \leqslant \infty$, *then* $u(t) < \underline{U}(t) + M_0\varepsilon^2$ *in* $J_\varepsilon := \left(c + \frac{3}{K}\varepsilon|\ln \varepsilon|, d - \frac{3}{K}\varepsilon|\ln \varepsilon|\right)$. *If* u *is a solution of* (1.4) *satisfying* $u > \sqrt{\frac{\lambda}{3}}$ *in* (c, d), *then* $u(t) > \bar{U}(t) - M_0\varepsilon^2$ *in* J_ε.

(ii) *If* $0 < \varepsilon \leqslant \varepsilon_0$ *and* u *is a bounded solution of* (1.4) *on* $(-\infty, \infty)$ *satisfying* $u < -\sqrt{\frac{\lambda}{3}}$ *in* (c, d) *with* $-\infty \leqslant c < d \leqslant \infty$, *then*

$$|u(t) - \underline{U}(t)| < M_0\varepsilon^2 \qquad \textit{for } t \in J_\varepsilon. \tag{4.1}$$

If instead u *satisfies* $u > \sqrt{\frac{\lambda}{3}}$ *over* (c, d), *then*

$$|u(t) - \bar{U}(t)| < M_0\varepsilon^2 \qquad \textit{for } t \in J_\varepsilon. \tag{4.2}$$

Proof. It is easy to check that there are $M_0 > 0$ and $\varepsilon_0 > 0$ such that if $0 < \varepsilon \leqslant \varepsilon_0$ and $U_2 = \underline{U} + (M_0 - 1)\varepsilon^2$, then, $\varepsilon^2 U_2'' < U_2^3 - \lambda U_2 + \cos t$ on $(-\infty, \infty)$. Assume for contradiction that there are $\varepsilon \in (0, \varepsilon_0)$ and $\hat{t} \in J_\varepsilon$ such that $u \geqslant \underline{U} + M_0\varepsilon^2$ at the point $\hat{t}$. Assume first that $u'(\hat{t}) \geqslant \underline{U}'(\hat{t})$. Let $w = u - U_2$. Since $\varepsilon^2 w'' > (u^2 + uU_2 + U_2^2 - \lambda)w$ in $[c, d]$, $w(\hat{t}) > 0$ and $w'(\hat{t}) \geqslant 0$, it follows that $w > 0$, $w' > 0$ and $\varepsilon^2 w'' > K^2 w$ for $t \in [\hat{t}, d]$. Then for $\hat{t} \leqslant t < s \leqslant d$ we find that

$$\begin{aligned} &\frac{d}{ds}\left(w(s) + \sqrt{w^2(s) - w^2(t) + \frac{\varepsilon^2}{K^2}(w')^2(t)}\right) \\ &\quad > \frac{K}{\varepsilon}\left(w(s) + \sqrt{w^2(s) - w^2(t) + \frac{\varepsilon^2}{K^2}(w')^2(t)}\right) \end{aligned}$$

and so for $\hat{t} \leqslant t < T \leqslant d$ we obtain

$$w(T) + \sqrt{w^2(T) - w^2(t) + \frac{\varepsilon^2}{K^2}(w')^2(t)} > \left[w(t) + \frac{\varepsilon}{K}\,w'(t)\right]e^{\frac{K}{\varepsilon}(T-t)}. \tag{4.3}$$

Assume that $d < \infty$. Evaluating (4.3) at $T = d$ and $t = \hat{t}$ and using $d - \hat{t} \geqslant \frac{3}{K}\varepsilon|\ln \varepsilon|$ gives

$$w(d) + \sqrt{w^2(d) - w^2(\hat{t}) + \frac{\varepsilon^2}{K^2}(w')^2(\hat{t})} > \left[w(\hat{t}) + \frac{\varepsilon}{K}w'(\hat{t})\right]\frac{1}{\varepsilon^3}.$$

Since $w(\hat{t}) \geqslant \varepsilon^2$ and $w'(\hat{t}) \geqslant 0$, it follows that the above inequality does not hold if ε is small.

Therefore, ε_0 can be chosen independent of $\hat{t}$ so that for $0 < \varepsilon \leqslant \varepsilon_0$, $u'(\hat{t}) < \underline{U}'(\hat{t})$ if $u(\hat{t}) \geqslant \underline{U} + M_0\varepsilon^2$. Then let $\hat{u}(t) = u(-t)$ and $\hat{U}_2(t) = U_2(-t)$. Apply the same argument for $\hat{u}$ as above to get the same contradiction if $\varepsilon < \varepsilon_0$.

If $d = \infty$, then letting $t \to \infty$, we see that the right side of (4.3) goes to ∞, contradicting the boundedness of the left side of (4.3). The other assertions of (i) can be proved similarly.

(ii) Recall that u_1 is the smallest bounded solution of (1.4) over $(-\infty, \infty)$ and $\underline{U}(0) < u_1 < \underline{U}(\pi)$. It follows from (i) that to prove (4.1) it suffices to show that $u_1(t) > \underline{U} - M_0\varepsilon^2$ on $(-\infty, \infty)$. Note that if ε is sufficiently small, then $\varepsilon^2 U_1'' > U_1^3 - \lambda U_1 + \cos t$, where $U_1 = \underline{U} - (M_0 - 1)\varepsilon^2$. Assume that there is an $\varepsilon \leqslant \varepsilon_0$ and $\hat{t} \in J_\varepsilon$ such that $u_1(\hat{t}) \leqslant \underline{U}(\hat{t}) - M_0\varepsilon^2$. Then, as above, by considering $(U_1 - u_1)(t)$ for $t > \hat{t}$ if $u_1'(\hat{t}) \leqslant U_1'(\hat{t})$ and for $t < \hat{t}$ if $u_1'(\hat{t}) > U_1'(\hat{t})$, we obtain the contradictions that $(U_1 - u_1)(t) \to \infty$ as $t \to \pm\infty$, respectively. This proves (4.1). (4.2) can be proved similarly. ∎

The next Lemma describes the asymptotic behavior of u_1, u_5 and their first-order derivatives as $\varepsilon \to 0$.

LEMMA 4.3. *Assume that $\lambda > \lambda_0$. If ε_0 is sufficiently small, then for $t \in (-\infty, \infty)$,*

$$|u_1(t) - \underline{U}(t)| < M_0\varepsilon^2 \qquad \textit{and} \qquad |u_5(t) - \bar{U}(t)| < M_0\varepsilon^2 \tag{4.4}$$

and

$$|u_1'(t) - \underline{U}'(t)| < M_0'\varepsilon \qquad \textit{and} \qquad |u_5'(t) - \bar{U}'(t)| < M_0'\varepsilon, \tag{4.5}$$

where M_0 and M_0' are constant, independent of ε.

Proof. The inequalities in (4.4) follow from (ii) of the above lemma. To show (4.5) we let $z = \frac{1}{\varepsilon^2}(u_1 - \underline{U})$, which satisfies

$$\varepsilon^2 z'' = (3\underline{U}^2 - \lambda)z + 3\varepsilon^2 \underline{U} z^2 + \varepsilon^4 z^3 - \underline{U}''. \tag{4.6}$$

Further, $z'(0) = 0$ and $|z(t)| < M_0$ for all $t \in (-\infty, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$. Multiply (4.6) by z' to get

$$\begin{aligned}\tfrac{1}{2}\varepsilon^2(z')^2(t) =& \tfrac{1}{2}(3\underline{U}^2(t) - \lambda)z^2(t) - \tfrac{1}{2}(3\underline{U}^2(0) - \lambda)z^2(0) - 3\int_0^t \underline{U}(s)\underline{U}'(s)z^2(s)\, ds \\ &+\varepsilon^2 \underline{U}(t)z^3(t) - \varepsilon^2 \underline{U}(0)z^3(0) - \int_0^t z^3(s)\underline{U}'(s)\, ds + \frac{1}{4}\varepsilon^4(z^4(t) - z^4(0)) \\ &- \underline{U}''(t)z(s) + \underline{U}''(0)z(0) + \int_0^t z(s)\underline{U}'''(s)\, ds. \end{aligned} \tag{4.7}$$

Since z, $\underline{U}$ and the derivatives of $\underline{U}$ are all bounded with the bounds independent of ε, it follows that the right-hand side of (4.7) is bounded by a constant, say, $\frac{1}{2}(M_0')^2$ over $[0, \pi]$, that is, $\varepsilon^2(z')^2(t) \leqslant (M_0')^2$ for $t \in [0, \pi]$ and $\varepsilon \in (0, \varepsilon_0]$. Hence (4.5) holds for $t \in [0, \pi]$. Since u_1 and $\underline{U}$ are 2π-periodic and even functions, we see that (4.5) holds for all $t \in (-\infty, \infty)$. ∎

LEMMA 4.4. *Assume that $\lambda > \lambda_0$. Then for each small $\mu > 0$ there is an $\varepsilon_\mu \in (0, 1)$ such that if $0 < \varepsilon < \varepsilon_\mu$ and u is a solution with $u > \sqrt{\frac{\lambda}{3}}$ in (c, d), then*

$$|u - u_5| + \frac{\varepsilon}{2K}|u' - u_5'| \leqslant M_1 e^{-\frac{K}{\varepsilon}\mu} \qquad \text{on } [c + \mu, d - \mu]. \tag{4.8}$$

If $u < -\sqrt{\frac{\lambda}{3}}$ in (c, d), then

$$|u - u_1| + \frac{\varepsilon}{2K}|u' - u_1'| \leqslant M_1 e^{-\frac{K}{\varepsilon}\mu} \qquad \text{on } [c + \mu, d - \mu].$$

Here K is defined in Lemma 4.2 and $M_1 = 2(\bar{U}(\pi) - \sqrt{\frac{\lambda}{3}})$.

Proof. (i) Let $w = u_5 - u$. Since $w > 0$ and $u > \sqrt{\frac{\lambda}{3}}$ in (c, d), it follows that $\varepsilon^2 w'' > K^2 w$ in $[c, d]$. From Lemma 4.2 we see that $w < M_0 \varepsilon^2$ in $[c + \mu, d - \mu]$.

We first assume that $w' \geqslant 0$ at $t_0 := c + \mu$. Then (4.3) holds for this w with $t_0 \leqslant t < T \leqslant d$, and so for $t_0 \leqslant t < T \leqslant d$,

$$w(t) + (1 - e^{-\frac{K}{\varepsilon}(T-t)})\frac{\varepsilon}{K}|w'(t)| \leqslant 2w(T)e^{-\frac{K}{\varepsilon}(T-t)}. \tag{4.9}$$

Since $w(T) < \bar{U}(\pi) - \sqrt{\frac{\lambda}{3}}$, and $e^{-\frac{K}{\varepsilon}\mu} < \frac{1}{2}$ if ε is small, it follows that for $t \in [c - \mu, d - \mu]$,

$$w(t) + \frac{\varepsilon}{2K}|w'(t)| \leqslant M_1 e^{-\frac{K}{\varepsilon}(T-t)}. \tag{4.10}$$

Let $T = d$. Since $d - t \geqslant \mu$ for $t \in [t_0, d - \mu]$, (4.8) follows immediately from (4.10) for $t \in [c + \mu, d - \mu]$. Assume that $w'(t_0) < 0$. Let $\hat{t} = \sup\{t \in [t_0, d - \mu] : w' < 0 \text{ in } [t_0, t)\}$. From what we just proved, we can assume that $\hat{t} = d - \mu$. Since w' is decreasing in $[c, d]$, $w' < 0$ in $[c - \mu, \hat{t}]$, and $w > 0$ in $[c, d]$, it suffices to show that (4.8) holds at $t = t_0$. Integrating $w''w' < \frac{K^2}{\varepsilon^2}ww'$ over $[c, t_0]$ gives

$$w(t_0) + \frac{\varepsilon}{K}|w'(t_0)| \leqslant \left[w(c) + \sqrt{w^2(c) - w^2(t_0) + \frac{\varepsilon^2}{K^2}(w')^2(t_0)}\right]e^{-\frac{K}{\varepsilon}\mu}$$

which implies that (4.8) holds at $t = t_0$. The proof of the inequality for u_1 is similar and therefore is omitted. ∎

From Lemmas 4.2–4.4 one immediately obtains a refinement of Lemma 4.2.

COROLLARY 4.5. *Assume that $\lambda > \lambda_0$. For any small $\mu > 0$, there is an $\varepsilon_\mu > 0$ such that if $0 < \varepsilon < \varepsilon_\mu$ and u is a bounded solution of (1.4) over $(-\infty, \infty)$ satisfying $u < -\sqrt{\frac{\lambda}{3}}$ over (c, d), then for $t \in [c + \mu, d - \mu]$*

$$|u(t) - \underline{U}(t)| + \varepsilon|u'(t) - \underline{U}'(t)| \leqslant (M_0 + M_0' + 1)\varepsilon^2. \tag{4.11}$$

If, instead, u satisfies $u > \sqrt{\frac{\lambda}{3}}$ over (c, d), then for $t \in [c + \mu, d - \mu]$,

$$|u(t) - \bar{U}(t)| + \varepsilon|u'(t) - \bar{U}'(t)| \leqslant (M_0 + M_0' + 1)\varepsilon^2. \tag{4.12}$$

The next lemma is a refinement of Lemma 4.1 if $\lambda > \lambda_0$.

LEMMA 4.6. *Assume that* $\lambda > \lambda_0$. *For any small* $\mu > 0$, *there is an* $\varepsilon_\mu > 0$ *such that if* $0 < \varepsilon < \varepsilon_\mu$ *and* u *is a bounded solution of* (1.4) *which satisfies* $u < -\sqrt{\lambda}$ *at some point in* $J_{k\mu} := \left[\frac{(2k-1)\pi}{2} + \frac{\mu}{2}, \frac{(2k+1)\pi}{2} - \frac{\mu}{2}\right]$ *for some even integer* k, *then* (4.11) *holds in* $M_{k\mu} := \left[\frac{(2k-1)\pi}{2} + \mu, \frac{(2k+1)\pi}{2} - \mu\right]$. *Similarly,* (4.12) *holds in* $M_{k\mu}$ *if* u *satisfies* $u > \sqrt{\lambda}$ *at some point in* $J_{k\mu}$ *for some odd integer* k.

Proof. To show the first part of lemma, from Corollary 4.5 it suffices to show that for ε small, $u < -\sqrt{\frac{\lambda}{3}}$ in $J_{k\mu}$. Assume that this is false. Then there is a sequence ε_n with $\lim_{n\to\infty} \varepsilon_n = 0$ and a sequence of $t_n \in J_{k\mu}$ with $\lim_{n\to\infty} t_n = \hat{t}$ for some $\hat{t} \in J_{k\mu}$ such that $u_n(t_n) = -\sqrt{\lambda}$. Then by a phase plane argument we obtain that u_n will reach b by time $\hat{t} + O(\varepsilon_n)$ as $n \to \infty$, contradicting that u_n is bounded. ∎

Since u_{p} is antisymmetric around $\pi/2$, from Lemma 4.6 we immediately get

THEOREM 4.7. *Assume that* $\lambda > \lambda_0$. *For any small* $\mu > 0$ *there is an* $\varepsilon_\mu > 0$ *such that if* $0 < \varepsilon < \varepsilon_\mu$, *then* (4.11) *and* (4.12) *hold for* $u = u_{\mathrm{p}}$ *in* $\left[-\frac{\pi}{2} + \mu, \frac{\pi}{2} - \mu\right]$ *and* $\left[\frac{\pi}{2} + \mu, \frac{3\pi}{2} - \mu\right]$, *respectively.*

The next theorem is about the asymptotic behavior of u_2 as $\varepsilon \to 0$.

THEOREM 4.8. (i) *Let* V_{-1} *be the homoclinic solution of* $\ddot{V} = V^3 - \lambda V - 1$ *such that* $\lim_{\tau\to\pm\infty} V_{-1}(\tau) = \underline{U}(\pi)$ *and* $\dot{V}_{-1}(0) = 0$. *Then* $\lim_{\varepsilon\to 0} u_2(\pi) = V_{-1}(0)$. *For any given* $T > 0, u_2(\pi + \varepsilon\tau) - V_{-1}(\tau)$ *approaches zero uniformly for* $\tau \in [-T, 0]$ *as* $\varepsilon \to 0$. (ii) *For any small* $\mu > 0$ *there is an* $\varepsilon_\mu > 0$ *such that if* $0 < \varepsilon < \varepsilon_\mu$, *then* (4.11) *holds in* $[0, \pi - \mu]$ *for* $u = u_2$.

Proof. Let $v(\tau) = u_2(\pi + \varepsilon\tau)$. Then v satisfies

$$\ddot{v} = v^3 - \lambda v + \cos(\pi + \varepsilon\tau), \qquad v(0) = u_2(\pi), \qquad \dot{v}(0) = 0.$$

Then (i) follows easily by a phase plane argument. To show (ii), we take T so large that $V_{-1}(-T) < -\sqrt{\frac{\lambda}{3}}$. Then from (i) it follows that if ε is sufficiently small, then $u_2(\pi - \varepsilon T) < -\sqrt{\frac{\lambda}{3}}$. Therefore, if ε is small enough to satisfy $\varepsilon T < \frac{\mu}{2}$, then, since $u_2' > 0$ in $(0, \pi)$, $u_2(t) \leqslant u_2(\pi - \frac{\mu}{2}) < -\sqrt{\frac{\lambda}{3}}$ for $t \in [-\frac{\mu}{2}, \pi - \frac{\mu}{2}]$. Hence (ii) follows from Corollary 4.5. ∎

4.2. Further Periodic Solutions

As pointed out earlier, the classical analysis of Duffing's equation, with results such as those in [NM], corresponded in a sense to taking λ large. These solutions are bounded as $\lambda \to \infty$. The solutions u_1, u_2, u_3 found in Section 2 all have minimum values (which are also their initial values) below $-\sqrt{\lambda}$, while u_4 and u_5 have maxima above $\sqrt{\lambda}$. In this section, we begin by finding solutions with oscillations, distributed reasonably evenly in $(0, 2\pi)$, and with initial values in the interval $(-\sqrt{\lambda}, 0)$. This is for a fixed λ and small ε. Then we consider the behavior of those solutions if we keep the number of oscillations fixed (together with λ) and let $\varepsilon \to 0$. It is found that the oscillations collect near odd multiples of $\frac{\pi}{2}$, forming internal "layers" between $\bar{U}$ and $\underline{U}$ (see Fig. 6). The result is related to one in [HM2] about a similar nonlinear forced oscillator (derived in [OOJ]), where the nonlinearity was quadratic rather than cubic. However the Duffing equation has a richer collection of solutions than the equation in [HM2].

A question of interest is to what extent the results in this paper depend on the symmetry of the cosine function. This is addressed further in Section 5, but the techniques in Theorems 4.9 and 4.12 of the current section do not use the symmetry of cosine around $\frac{\pi}{2}$. These should apply to more general forcing functions, as we will discuss in future work. On the other hand, the proof of Theorem 4.14 below shows how use of the symmetry around $\frac{\pi}{2}$ can

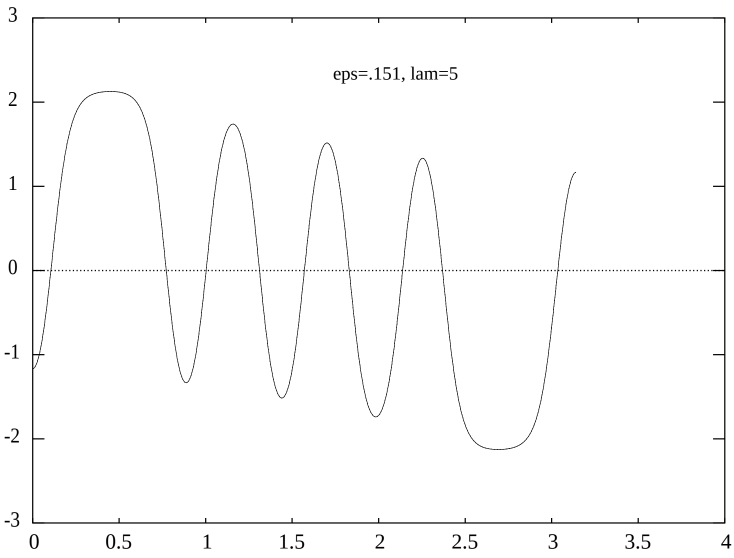

FIGURE 6

greatly simplify some proofs. The key fact is that if u is a solution of (1.4) such that $u\left(\frac{\pi}{2}\right) = 0$, then $u\left(\frac{\pi}{2} - t\right) = -u\left(\frac{\pi}{2} + t\right)$ for all t.

In some of the following results we will switch back and forth between the original scaling and that in the slow equation (3.2), with $t = \varepsilon\tau$, $u(t) = v(\tau)$. For any $\kappa \in (0, 1]$, the equation

$$\ddot{V} = V^3 - \lambda V + \kappa \tag{4.13}$$

has a unique solution, forming the homoclinic orbit, with $V(\pm\infty) = \bar{U}(\arccos\kappa)$ and $\dot{V}(0) = 0$. We denote this solution by V_κ. Let

$$\Lambda = \inf\{\lambda > \lambda_0 : V_1(0) < U_0(\pi)\}.$$

Thus, for $\lambda > \Lambda$, the homoclinic orbit at $\tau = 0$ $(\kappa = 1)$ has a minimum below $U_0(\pi)$. It is easy to show that $\Lambda < 3$.

THEOREM 4.9. *Suppose that $\lambda > \Lambda$. Then for any $N > 0$ there is an ε_0 such that if $0 < \varepsilon < \varepsilon_0$, and if m is any positive integer with $m \leqslant N$, then* (1.4) *has a periodic solution with exactly m local maxima in $(0, \pi)$, the last being at π. Also, these periodic solutions have a local minimum at $t = 0$ and satisfy $-\sqrt{\lambda} < u(0) < U_0(\pi)$ and $U_0(\pi) < u(\pi) < \bar{U}(\pi)$.*

Proof. We will need the following lemma, which is similar to Theorem 2 in [HM2].

LEMMA 4.10. *Let u be a solution of* (1.4) *with $u'(0) = 0$. If t_1 and t_2 are two successive minima (maxima) of u with $0 < t_1 < t_2 < \pi$, then $u(t_2) < u(t_1)$.*

Proof. We only show the case that $u(t_1)$ and $u(t_2)$ are successive maxima of u and the other case can be proved similarly. Assume that u reaches its maximum between t_1 and t_2 at $t = t_m$. Then multiply (1.4) by u' and integrate over $[t_-, t_+]$, where $t_1 < t_- < t_m < t_+ < t_2$ and $u(t_-) = u(t_+)$, to yield

$$\begin{aligned}\frac{1}{2}\varepsilon^2[(u')^2(t_+) - (u')^2(t_-)] &= \int_{t_-}^{t_+} u'(t)\cos t\,dt = \left(\int_{t_-}^{t_m} + \int_{t_m}^{t_+}\right) u'(t)\cos t\,dt \\ &= \int_{u(t_-)}^{u(t_m)} (\cos t_-(u) - \cos t_+(u))\,du > 0.\end{aligned}$$

Here we use that $\cos t$ is decreasing in $(0, \pi)$, and $t_-(u)$ and $t_+(u)$ are the inverse functions of $u = u(t)$ for $t \in [t_1, t_m]$ and $t \in [t_m, t_2]$, respectively. It then follows that $u'(t_-) = 0$ before $u'(t_+) = 0$, which implies that $u(t_1) > u(t_2)$ as required. ∎

For $\lambda > \lambda_0$, recall that $\underline{U}(t) < U_0(t) < \bar{U}(t)$ are the three solutions of $u^3(t) - \lambda u(t) + \cos t = 0$. For given $\lambda > \Lambda$, let $\delta_0 = \bar{U}_0(\pi) - V_1(0)$. We also need the fact, easily proved, that if V_κ is the homoclinic solution of (2.1) defined earlier, for $0 < \kappa \leqslant 1$, then the minimum values $V_\kappa(0)$ are decreasing with respect to κ.

Also, for each α, let v_α be the solution of (3.2) such that $v(0) = \alpha$, $\dot{v}(0) = 0$. Let $\hat{\alpha} = U_0(\pi) - \frac{\delta_0}{2}$.

Choose $N > 0$. When $\varepsilon = 0$, $v_{\hat{\alpha}}$ is periodic, and so has $N + 2$ local maxima in some interval $[0, T_N]$. For sufficiently small ε, $v_{\hat{\alpha}}$ still has at least $N + 1$ local maxima in $\left(0, \frac{\pi}{\varepsilon}\right)$. Hence, $u_{\hat{\alpha}}$ has at least $N + 1$ maxima in $(0, \pi)$. Suppose that these are at $0 < t_1 < t_2 < \cdots < t_{N+1} < \pi$. Further, from Lemma 4.10 we see that $u_{\hat{\alpha}}(t_{i+1}) < u_{\hat{\alpha}}(t_i)$ for $0 < i \leqslant N$.

If any of the maxima t_i for $i < N$ are such that $u_{\hat{\alpha}}(t_i) < \underline{U}(t_i)$, then because $\underline{U}$ is increasing in $(0, \pi)$, we must have $u < \underline{U}$ in (t_i, π) (if the solution exists out to π), and so $u_{\hat{\alpha}}$ could not have any more maxima. This shows that for $1 \leqslant i \leqslant N$, the maxima of $u_{\hat{\alpha}}$ must lie in the interval $[U_0(t), \bar{U}(t)]$. However, if $u' = u'' = 0$ at some t in $(0, \pi)$, then $u''' = -\sin(t) < 0$, so this could not be a maximum. Hence, the first N maxima of $u_{\hat{\alpha}}$ must lie in $(U_0(t), \bar{U}(t))$.

The solution $u_{\hat{\alpha}}$ must have a minimum at $s_0 = 0$ and further minima $s_1, \ldots, s_N \in (0, \pi)$. Also $\hat{\alpha} > u(s_1) > u(s_2) > \cdots > u(s_N)$, by Lemma 4.10. Therefore, $u_{\hat{\alpha}}(t_i) \geqslant u_{\hat{\alpha}}(s_{i-1}) + \frac{\delta_0}{2} > u_{\hat{\alpha}}(s_i) + \frac{\delta_0}{2}$ for $i = 1, \ldots, N$.

Now decrease α from $\hat{\alpha}$. Since $u'' < 0$ at a maximum, the maxima are continuous in α, and so remain above U_0 as long as they exist. If $\alpha = -b$, u_α has no local minima. But as long as there are, say, $M \leqslant N$ local maxima in $(0, \pi)$ with $u > U_0$ and with $\alpha < \hat{\alpha}$ these maxima and their intervening minima are separated, in that $u_\alpha(t_i(\alpha)) - u_\alpha(s_i(\alpha)) \geqslant \frac{\delta_0}{2}$ for $1 \leqslant i \leqslant M$. This means that the number of maxima cannot decrease until one crosses $t = \pi$. This must happen successively for each maximum, which proves Theorem 4.9. ∎

Remark. 4.11. With further estimates, it can be shown that there is a constant $K > 0$ such that if ε is sufficiently small, then for each positive integer $m \leqslant \frac{K}{\varepsilon}$, (1.4) has at least one 2π-periodic solution which has exactly m maxima in $(0, \pi]$.

In the next theorem we shall describe the asymptotic behavior, as $\varepsilon \to 0$, of 2π-periodic solutions of (1.4) with m maxima in $(0, \pi]$ and with $u(0) < 0$, $u(\pi) > 0$. The result, informally, is that if m is kept fixed as $\varepsilon \to 0$ then the internal maxima and minima all tend to $\frac{\pi}{2}$, while near 0 and π the solutions tend to homoclinic orbits of the appropriate limiting equation. Thus, these solutions have a single "spike" at both 0 and π, and internal layers where they are close to appropriate heteroclinic orbits, near $\frac{\pi}{2}$ (all lying in an $O(\varepsilon|\ln \varepsilon|)$ neighborhood of $\frac{\pi}{2}$). In Theorem 4.14, we obtain 2π-periodic solutions without spikes at 0 and π, using the symmetry of the cosine. In forthcoming work, we expect to consider other multilayer solutions, combining spike and nonspike behaviors at multiples of π, without reliance on symmetry.

Recall that V_1 is the homoclinic solution of $\ddot{v} = v^3 - \lambda v + 1$ such that $\lim_{\tau\to\pm\infty} v(\tau) = \bar{U}(0)$ and $\dot{v}(0) = 0$. Let $V_{-1}(t) = V_1(-t)$. Let V_{0^+} be the heteroclinic solution of $\ddot{v} = v^3 - \lambda v$ with $\lim_{\tau\to-\infty} v(\tau) = \bar{U}\left(\frac{\pi}{2}\right)$, $\lim_{\tau\to\infty} v(\tau) = \underline{U}\left(\frac{\pi}{2}\right)$ and $v(0) = \bar{U}\left(\frac{\pi}{2}\right) - \delta$, where δ is any fixed positive number satisfying $\delta < \bar{U}(0) - \sqrt{\frac{\lambda}{3}}$. Also, let $V_{0^-}(t) = -V_{0^+}(-t)$.

THEOREM 4.12. *Suppose that* $\lambda > \Lambda$. *For an integer* $N \geqslant 2$ *choose* ε_0 *as in Theorem* 4.9. *Let m be an integer with* $1 \leqslant m \leqslant N$, *and for each* $\varepsilon \in (0, \varepsilon_0)$ *let* u_ε *be a* 2π*-periodic solution of* (1.4) *satisfying* $u_\varepsilon'(0) = u_\varepsilon'(\pi) = 0$ *and* $-\sqrt{\lambda} < u_\varepsilon(0) < U_0(0)$, *and having exactly m maxima in* $(0, \pi]$. *Let* $t_1, \ldots, t_m$ *and* $s_1, \ldots, s_{m-1}$, *which all might depend on* ε, *be the maxima and the minima of* u_ε *in* $(0, \pi]$, *respectively, such that* $0 < t_1 < s_1 < t_2 < s_2 < \cdots < t_{m-1} < s_{m-1} < t_m = \pi$. *Then, as* $\varepsilon \to 0$,

(i) $u_\varepsilon(0) \to V_1(0)$, $u_\varepsilon(\pi) \to V_{-1}(0)$.

(ii) *for each* $1 \leqslant j \leqslant m-1$, $t_j \to \frac{\pi}{2}$, $s_j \to \frac{\pi}{2}$, $u_\varepsilon(t_j) \to \bar{U}\left(\frac{\pi}{2}\right)$, *and* $u_\varepsilon(s_j) \to \underline{U}\left(\frac{\pi}{2}\right)$.

(iii) *for any given* $T > 0$, $|u_\varepsilon(\varepsilon\tau) - V_1(\tau)| + |\varepsilon u_\varepsilon'(\varepsilon\tau) - \dot{V}_1(\tau)| \to 0$ *uniformly for* $\tau \in [0, T]$, *and* $|u_\varepsilon(\pi + \varepsilon\tau) - V_{-1}(\tau)| + |\varepsilon u_\varepsilon'(\pi + \varepsilon\tau) - \dot{V}_{-1}(\tau)| \to 0$ *uniformly for* $\tau \in [-T, 0]$.

Also, for $1 \leqslant j \leqslant m-1$, *let*

$$T_j = \sup\{t \in (t_j, \pi) : u_\varepsilon > u_5 - \delta \ \textit{in} \ (t_j, t]\}$$

and

$$S_j = \sup\{t \in (s_j, \pi) : u_\varepsilon < u_1 + \delta \ \textit{in} \ (s_j, t]\}.$$

Then $T_j - t_j < \frac{3\varepsilon}{K}|\ln\varepsilon|$ *and* $S_j - s_j < \frac{3\varepsilon}{K}|\ln\varepsilon|$, *where K is defined in Lemma* 4.2. *Further, for any given* $T > 0$, *as* $\varepsilon \to 0$, $|u_\varepsilon(T_j + \varepsilon\tau) - V_{0^+}(\tau)| + |\varepsilon u_\varepsilon'(T_j + \varepsilon\tau) - \dot{V}_{0^+}(\tau)| \to 0$ *and* $|u_\varepsilon(S_j + \varepsilon\tau) + V_{0^-}(\tau)| + |\varepsilon u_\varepsilon'(T_j + \varepsilon\tau) + \dot{V}_{0^-}(\tau)| \to 0$, *uniformly for* $\tau \in [0, T]$.

Finally, for any small $\mu > 0$ *there is an* $\varepsilon_\mu > 0$ *such that if* $0 < \varepsilon < \varepsilon_\mu$, *then* (4.12) *and* (4.11) *hold in* $\left[\mu, \frac{\pi}{2} - \mu\right]$ *and* $\left[\frac{\pi}{2} + \mu, \pi - \mu\right]$, *respectively.*

Proof. In the proof we shall suppress the dependence of u on ε. We first show that $\lim_{\varepsilon\to 0} u(0) = V_1(0)$. Assume that this is false. Since $u(0)$ is bounded, there is a sequence ε_n with $\lim_{n\to\infty} \varepsilon_n = 0$ such that as $n \to \infty$, the corresponding $u(0)$ approaches $\beta_0 \in [-\sqrt{\lambda}, U_0(0)]$. For simplicity, we assume that $\lim_{\varepsilon\to 0} u(0) = \beta_0$. Let $z(\tau) = u(\varepsilon\tau)$. Then z satisfies

$$\ddot{z} = z^3 - \lambda z + \cos \varepsilon\tau, \qquad z(0) = u(0), \qquad \dot{z}(0) = 0.$$

Let z_0 be the solution of

$$\ddot{z}_0 = z_0^3 - \lambda z_0 + 1, \qquad z_0(0) = \beta_0, \qquad \dot{z}_0(\tau) = 0.$$

Since $\lim_{\varepsilon\to 0} z(0) = \beta_0$, it follows from the continuity of solutions with respect to parameters that for any given $T > 0$ if z_0 is defined on $[0, T]$, then $\lim_{\varepsilon\to 0} z(\tau) = z_0(\tau)$ uniformly for $\tau \in [0, T]$.

Assume that $\beta_0 < V_1(0)$. Then there is a $T_0 > 0$ such that $\dot{z}_0(\tau) > 0$ for $\tau \in (0, T_0)$ and $\lim_{\tau\to T_0^-} z_0(\tau) = \infty$. It follows that for ε sufficiently small z crosses b, which is impossible. Hence, $\beta_1 > V_1(0)$. Then z_0 is a periodic function with a period $\tilde{T}_0 > 0$ and so z_0 has $m+1$ maxima in the interval $(0, (m+2)\tilde{T}_0)$. Hence by continuity, for ε sufficiently small, $z(\tau)$ also has $m+1$ maxima in the interval $(0, (m+2)\tilde{T}_0)$, which implies that u has $m+1$ maxima in $(0, \varepsilon(m+2)\tilde{T}_0) \subset (0, \pi]$ for ε sufficiently small, contradicting the assumption on u. Therefore, $\lim_{\varepsilon\to 0} u(0) = V_1(0)$ and so (iii) and the first part of (i) follow. The rest of (i) and (iii) can be proved similarly.

We next show that $\lim_{\varepsilon\to 0} t_1 = \frac{\pi}{2}$. Suppose not. Since $t_1 \in (0, \pi]$, there is a sequence ε_n with $\lim_{n\to\infty} \varepsilon_n = 0$ such that $\lim_{n\to\infty} t_1 = \bar{t} \neq \frac{\pi}{2}$ for some $\bar{t} \in [0, \pi]$. Again, for simplicity, we assume that $\lim_{\varepsilon\to 0} t_1 = \bar{t}$.

Since $u(\pi) \to V_{-1}(0)$ as $\varepsilon \to 0$, it follows from Lemma 4.10 that $\bar{t} \neq \pi$. We suppose now that $\frac{\pi}{2} < \bar{t} < \pi$. Let $z(\tau) = u(t_1 + \varepsilon\tau)$. Then z satisfies

$$\ddot{z} = z^3 - \lambda z + \cos(t_1 + \varepsilon\tau), \qquad z(0) = u(t_1), \qquad \dot{z}(0) = 0.$$

For $b = b(\lambda) = \sqrt{\lambda + \frac{1}{2\lambda}}$ (as is defined in Lemma 3.7), let

$$T_b = \int_{-b}^{u(t_1)} \frac{dy}{\sqrt{\frac{1}{2}(y^4 - u(t_1)^4) - \lambda(y^2 - u(t_1)^2) + 2(y - u(t_1))\cos t_1}},$$

and

$$T = \sup\{\tau \in (0, T_b + 1) : z > -b,\ \dot{z} < 0 \text{ in } (0, \tau)\}.$$

Observe that $\bar{U}(t_1) > z(0) = u(t_1) > \bar{U}(t_1) - \delta$ if ε is sufficiently small. Hence, T_b is defined and $T_b < \infty$; $\ddot{z}(0) > 0$ and so T is well defined. Assume that ε is so small that $t_1 + \varepsilon(T_b + 1) \leqslant \pi$. Then on $(0, T)$ we have $\ddot{z} \leqslant z^3 - \lambda z + \cos t_1$ and so $(\dot{z})^2 > \frac{1}{2}(z^4 - z^4(0)) - \lambda(z^2 - z^2(0)) + 2(z - z(0))\cos t_1$. We see that $T < T_b$ and $\dot{z}(T) < 0$. Therefore, by the definition of T, it follows that $u(t_1 + \varepsilon T) = z(T) = -b$, which is impossible since $u(\tau) > -b$ for all τ.

We assume now that $\bar{t} \in [0, \frac{\pi}{2})$. Note that $u(t_1) < \bar{U}(t_1)$. We shall show that $\lim_{\varepsilon\to 0} u(t_1) = \bar{U}(\bar{t})$. For if this is false, then the boundedness of $u(t_1)$ implies that there is a sequence ε_n, with $\varepsilon_n \to 0$ as $n \to \infty$, such that $\lim_{n\to\infty} u(t_1) < \bar{U}(\bar{t})$. It then follows by arguments similar to those above that u has more than m maxima in $(0, \pi]$ for large n, which is a contradiction.

We now claim that $|T_1 - t_1| < \frac{3\varepsilon}{K}|\ln \varepsilon|$ for $\varepsilon > 0$ sufficiently small, where T_1 is defined in the statement of this theorem (4.12). Assume that this is false. Let $w = u_5 - u$. Then in (t_1, T_1), $w > 0$ and $\varepsilon^2 w'' \geqslant K^2 w$. Since $u' < 0$ and $u_5' > 0$ just to the right of t_1, we see that $w' > 0$ and $w''w' > (K^2/\varepsilon^2)ww'$ in $(t_1, T_1]$. Then, as in (4.3) (where w was slightly different), we again get

$$w(T) + \sqrt{w^2(T) - w^2(t) + \frac{\varepsilon^2}{K^2}(w')^2(t)} > \left[w(t) + \frac{\varepsilon}{K}w'(t)\right]e^{\frac{K}{\varepsilon}(T-t)} \tag{4.14}$$

for any $t_1 \leqslant t < T \leqslant T_1$.

We assume now that $\bar{t} > 0$. Since $\lim_{\varepsilon\to 0} \bar{U}'(t_1) = \bar{U}'(\bar{t}) = \frac{\sin \bar{t}}{3\bar{U}^2(\bar{t}) - \lambda}$, we use (4.5) to obtain

$$w'(t_1) = u_5'(t_1) \geqslant \bar{U}'(t_1) - M_0'\varepsilon \geqslant \frac{1}{2}\bar{U}'(\bar{t}) - M_0'\varepsilon > \frac{\sin \bar{t}}{4(3\bar{U}^2(\bar{t}) - \lambda)} > 0.$$

This estimate gives a lower bound on $w'(t_1)$ as ε tends to zero. Evaluating (4.14) at $t = t_1$ and $T = t_1 + \frac{3\varepsilon}{K}|\ln \varepsilon|$ and letting $\varepsilon \to 0$ shows that the right side of (4.14) goes to infinity, while the left side is bounded. This contradiction proves the claim if $\bar{t} \in (0, \frac{\pi}{2})$.

We now assume that $\bar{t} = 0$. We observe that

$$\begin{aligned} w'\left(t_1 + \frac{\varepsilon}{K}|\ln \varepsilon|\right) &> u_5'\left(t_1 + \frac{\varepsilon}{K}|\ln \varepsilon|\right) \geqslant \frac{1}{2}\bar{U}'\left(t_1 + \frac{\varepsilon}{K}|\ln \varepsilon|\right) - M_0'\varepsilon \\ &> \frac{\sin\left(\frac{\varepsilon}{K}|\ln \varepsilon|\right)}{4(3\bar{U}^2(0) - \lambda)} > \frac{\frac{\varepsilon}{K}|\ln \varepsilon|}{2\pi(3\bar{U}^2(0) - \lambda)}. \end{aligned}$$

Then evaluating (4.14) at $t = t_1 + \frac{\varepsilon}{K}|\ln \varepsilon|$ and $T = t_1 + \frac{3\varepsilon}{K}|\ln \varepsilon|$ and letting $\varepsilon \to 0$, the same contradiction will be obtained. This shows that $|T_1 - t_1| < \frac{3\varepsilon}{K}|\ln \varepsilon|$, as claimed. In particular, $T_1 - t_1 \to 0$ and so $T_1 \to \bar{t}$.

Continuing with our proof that $\bar{t} = \frac{\pi}{2}$, and under the assumption that this is false and $\bar{t} \in [0, \frac{\pi}{2})$, let $z(\tau) = u(T_1 + \varepsilon\tau)$. Then z satisfies

$$\begin{aligned} \ddot{z} = z^3 - \lambda z + \cos(T_1 + \varepsilon\tau), \qquad & z(0) = u(T_1) = u(T_1) - \delta, \\ \dot{z}(0) = \varepsilon u'(T_1). & \end{aligned} \tag{4.15}$$

We shall show that $\lim_{\varepsilon\to 0} \dot{z}(0) = \dot{z}_0(0)$, where z_0 is the homoclinic solution of

$$\ddot{z}_0 = z_0^3 - \lambda z_0 + \cos(\bar{t}) \tag{4.16}$$

with $z_0(0) = \bar{U}(\bar{t}) - \delta$ and $\dot{z}_0(0) < 0$.

Suppose that $\lim_{\varepsilon\to 0} \dot{z}(0) \neq \dot{z}_0(0)$. Since $\dot{z}(0) < 0$ and $\dot{z}(0)$ is bounded, which is easily verified, there is a sequence of values of ε, and corresponding solutions u with corresponding z, such that $\dot{z}(0)$ approaches a number $\sigma \neq \dot{z}_0(0)$ with $\sigma \leqslant 0$. We first assume that $\sigma > \dot{z}_0(0)$. Let z_1 be the periodic solution, with period $\tilde{T}$, to

$$\ddot{z}_1 = z_1^3 - \lambda z_1 + \cos(\bar{t}), \quad z_1(0) = \bar{U}(\bar{t}) - \delta, \qquad \dot{z}_1(0) = \sigma. \tag{4.17}$$

Since $T_1 \to \bar{t}$, $z(\tau)$ approaches $z_1(\tau)$ as $\varepsilon \to 0$ uniformly in compact intervals of τ, and it follows that for sufficient small ε, z oscillates more than $m+1$ times in $[0, (m+2)\tilde{T}]$, and so u has more than m maxima in $[0, \pi]$, a contradiction.

Hence, we can assume that $\sigma < \dot{z}_0(0)$ and again assume that z_1 solves (4.17). Then there is a $\hat{T} > 0$ such that $\dot{z}_1(\tau) < 0$ for $\tau \in [\hat{T}, 0]$ and $z_1(\hat{T}) > b$, which implies that for ε sufficiently small, $u(t_1 + \varepsilon\hat{T}) > b$, again a contradiction.

Therefore, as $\varepsilon \to 0$, $\dot{z}(0) \to \dot{z}_0(0)$ and z goes to z_0 uniformly in any compact interval. Since z_0 is homoclinic to $\bar{U}(\bar{t})$, it follows from continuity that after T_1, u will return to any given neighborhood of $\bar{U}(\bar{t})$ before $t = T_1 + M\varepsilon$ for some M independent of ε. Then, since $m = 2$, u increases and stays close to $\bar{U}$ till π, which implies that $\lim_{\varepsilon\to 0} u(\pi) = \bar{U}(\pi)$ and so $u(t_2) = u(\pi) > u(t_1)$ for ε small, contradicting Lemma 4.10. This shows that $\bar{t} = \frac{\pi}{2}$ for $m = 2$.

Then, z_0 is the heteroclinic solution connecting $\bar{U}\left(\frac{\pi}{2}\right)$ as $t \to -\infty$ to $\underline{U}\left(\frac{\pi}{2}\right)$ as $t \to \infty$. So, as $\varepsilon \to 0$ the point $(t, u(t))$ reaches a point as close to $\left(\frac{\pi}{2}, \underline{U}\left(\frac{\pi}{2}\right)\right)$ as we like, and then Lemma 4.10 implies that $\lim_{\varepsilon\to 0} s_1 = \frac{\pi}{2}$ and $\lim_{\varepsilon\to 0} u(s_1) = \underline{U}\left(\frac{\pi}{2}\right)$. For $t \in [t_1, \pi]$, $u(t) < u(t_1)$, again by Lemma 4.10. Further, $u(t)$ remains close to $\underline{U}(t)$ until t is close to π, for otherwise there would be similar contradictions to those obtained above. Hence, u must, after its last minimum, follow $\underline{U}$ until close to π and then, since its last maximum is above $U_0(\pi)$, follow V_{-1}. The bound on $S_j - s_j$ follows in the same way as the bound for $T_j - t_j$. For the final statement in the theorem, we use (i)–(iii) and Corollary 4.5. This proves the theorem for $m = 2$. The proof for $m > 2$ is similar. ∎

The solutions discussed in Theorem 4.12 have upward and downward pointing "spikes" at each multiple of π, corresponding to homoclinic orbits in the phase plane, as well as "layers" at odd multiples of $\frac{\pi}{2}$ corresponding to heteroclinic orbits. In the next theorem, we use the symmetry of cosine around $\frac{\pi}{2}$ to give a quick proof that there are also solutions with layers but without the spikes. In our future work we expect to show that these solutions exist without reliance on symmetry, but it is also valuable, we believe, to show how quickly a proof can be obtained in the symmetric case. We need a preliminary lemma.

LEMMA 4.13. *Assume that $\lambda > \lambda_0$. Let u be a solution of* (1.4) *with $u\left(\frac{\pi}{2}\right) = 0$ and $u'\left(\frac{\pi}{2}\right) = \beta$. Let $H(u) = \lambda u^2 - u^4/2$.*

(i) *If $\beta < -\frac{\lambda}{\sqrt{2}\varepsilon}$, then $u' < 0$ on $\left[\frac{\pi}{2}, \pi\right]$ and as long as u exists.*

(ii) *Assume that $\beta < -\frac{\sqrt{H(\underline{U}(\pi))}}{\varepsilon}$. If $t_1 \in \left(\frac{\pi}{2}, \pi\right)$ is the first time that $u' = 0$* (*if such a point exists*), *then $u(t_1) < \underline{U}(\pi)$.*

Proof. From (1.4) we have

$$\varepsilon^2(u')^2(t) = \varepsilon^2(u')^2\left(\frac{\pi}{2}\right) - H(u) + 2\int_{\frac{\pi}{2}}^{t} u' \cos s \, ds. \tag{4.18}$$

(i) Let $t_0 = \sup\{t \in \left(\frac{\pi}{2}, \pi\right) : u' < 0 \text{ and } u > -\sqrt{\lambda} \text{ in } \left(\frac{\pi}{2}, t\right)\}$. It follows from (4.18) that for $t \in \left(\frac{\pi}{2}, t_0\right), \varepsilon^2(u')^2(t) > \varepsilon^2\beta^2 - H(u) > \varepsilon^2\beta^2 - \frac{\lambda^2}{2} > 0$, from which the assertion in (i) follows.

(ii) From (1.4), we first have $-\sqrt{\lambda} < u(t_1) < 0$. Evaluate (4.18) at $t = t_1$ to give

$$0 = \varepsilon^2\beta^2 - H(u(t_1)) + 2\int_{\frac{\pi}{2}}^{t_1} u' \cos(s) \, ds$$

so that $H(u(t_1)) > \varepsilon^2\beta^2$. This implies that $u(t_1) < \underline{U}(\pi)$, as desired. ∎

THEOREM 4.14. *For any given integer m, there is an $\varepsilon_m > 0$ such that for $\varepsilon \in (0, \varepsilon_m)$,* (1.4) *has a solution u such that $u'(0) = u'(\pi) = 0$, $u\left(\frac{\pi}{2}\right) = 0$, $u'\left(\frac{\pi}{2}\right) < 0$, and u has m minima and m maxima in $\left[\frac{\pi}{2}, \pi\right]$. If we denote these minima* *and maxima by* s_j *and* t_j, *respectively, with* $1 \leqslant j \leqslant m$, *then*

$$\frac{\pi}{2} < s_1 < t_1 < s_2 < t_2 < \cdots < s_m < t_m = \pi,$$

$$\lim_{\varepsilon\to 0} s_1 = \cdots = \lim_{\varepsilon\to 0}{>}\, s_m = \frac{\pi}{2}, \qquad \lim_{\varepsilon\to 0} u(s_1) = \cdots = \lim_{\varepsilon\to 0} u(s_m) = \underline{U}\Big(\frac{\pi}{2}\Big),$$

$$\lim_{\varepsilon\to 0} t_1 = \cdots = \lim_{\varepsilon\to 0} t_{m-1} = \frac{\pi}{2}, \qquad \lim_{\varepsilon\to 0} u(t_1) = \cdots = \lim_{\varepsilon\to 0} u(t_{m-1}) = \bar{U}\Big(\frac{\pi}{2}\Big),$$

$$u'(t) > 0, \qquad \underline{U}\Big(\frac{\pi}{2}\Big) < u(t) < \underline{U}(\pi) \qquad \textit{for } s_m < t < \pi.$$

On any compact subintervals of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *and* $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, u *approaches* $\bar{U}$ *and* $\underline{U}$ *uniformly as* $\varepsilon \to 0$.

Proof. Let u_β denote the solution u of (1.4) satisfying $u\left(\frac{\pi}{2}\right) = 0$ and $u'(\pi) = \beta$. Choose $\beta_1 \in \left(-\frac{\lambda}{\sqrt{2}\varepsilon}, -\frac{\sqrt{H(\underline{U}(\pi))}}{\varepsilon}\right)$ and $\beta_2 < -\frac{\lambda}{\sqrt{2}\varepsilon}$. By a phase plane argument, there is an ε_m such that for $\varepsilon < \varepsilon_m$, the solution u_{β_1} has at least m minima and m maxima in $\left(\frac{\pi}{2}, \pi\right)$ with all its first m minima lying between $\underline{U}$ and U_0 and all its first m maxima lying between U_0 and $\bar{U}$. Denote the mth minimum and maximum by s_m and t_m. We consider the change in these minima as β decreases from β_1. Part (ii) of Lemma 4.13 shows that all the minima in $(\frac{\pi}{2}, \pi]$ lie below the line $u = \underline{U}(\pi)$, and so they can neither pass through $t = \pi$ nor disappear at the middle branch U_0 of $u'' = 0$. Since u_{β_2} does not have any minimum at all, it follows that as we decrease β_1 to β_2, all the minima will disappear by crossing the lower branch of $u'' = 0$ before $t = \pi$.

Let $\beta_3 = \inf\{\bar{\beta} | s_m$ is defined continuously for $\beta \in (\bar{\beta}, \beta_1)$ as the mth minimum after $\frac{\pi}{2}\}$. Then $u'_{\beta_3}(s_m) = 0$ and $u_{\beta_3}(s_m) = \underline{U}(s_m)$. For $\beta - \beta_3 > 0$ small, the mth maximum $t_m(\beta)$ of u_β exists. Clearly $\lim_{\beta\to\beta_3} t_m(\beta) = s_m(\beta_3)$. As before, if $u_\beta(t_m) = \underline{U}(t_m)$, we get $u'_\beta(t) < 0$ for all $t - t_m$ small, contradicting the definition of t_m. Hence, $u_\beta(t_m(\beta)) < \underline{U}(t_m(\beta))$. Therefore, as we raise β from β_3, $t_m(\beta)$ has to move toward to π under the lower branch $\underline{U}$ of $u'' = 0$. This maximum $t_m(\beta)$ cannot disappear by merger with another minimum in $\left[\frac{\pi}{2}, \pi\right]$, since this minimum would have to lie above $u_\beta(s_m)$, contradicting Lemma 4.10. Since $u_{\beta_1}(t_m) > U_0(\pi) > \underline{U}(\pi)$, there must be a point of discontinuity of the mth maximum before reaching β_1, and so there is a $\beta_4 \in (\beta_3, \beta_1)$ such that $t_m(\beta)$ is continuous in $[\beta_3, \beta_4]$, $t_m(\beta_4) = \pi$ and $u'_{\beta_4}(t_m(\beta_4)) = 0$. Then u_{β_4} is the desired solution. The asymptotic formulas stated in this theorem can be proved in a similar way to that of Theorem 4.12. ∎

The existence of these solutions, and many more, with different numbers of oscillations between different zeros of $\cos t$, was conjectured independently by H. Matano (private communication). This was for a more general class of equations, but the result was limited to a finite interval, so that "chaos" was not involved. Similar solutions were obtained for a different equation by Nakashima [NAK]. We wish to thank Professor Matano for helpful correspondence, and in particular for sharing his conjecture with us. While the results in this paper, about periodic solutions with multiple internal layers, were obtained independently, our thoughts concerning nonperiodic solutions with more than three internal layers near odd multiples of $\frac{\pi}{2}$ were previously somewhat vague, and we have been inspired to pursue this topic further by Professor Matano's conjecture. We hope in future work to include these solutions, and a correspondingly richer result about symbolic dynamics. The methods proposed by Matano for obtaining these solutions are very different from ours.

Nakashima, in [NAK], has also studied the stability of the oscillating solutions for her equation, including the dimensions of the unstable manifolds (Morse Index). This has implications for the dimension of the global attractor for the problem (1.2)–(1.3). Results of this type were also obtained in [HM2], for a different equation, though they were not stated in these terms.

4.3. Isolation and Stability of u_{p}

In this section, we show how a shooting method can give results similar to those in [AMPP] for (1.4), without the use of infinite-dimensional analysis or abstract dynamical systems. Using standard ode methods we prove the linearized stability of the three solutions found in Section 1 with respect to (1.2)–(1.4), and the isolation of u_{p} from other solutions satisfying $u'(0) = u'(k\pi) = 0$. Full nonlinear stability follows by standard methods, laid out explicitly for this problem in [BF]. However, stability is inherently an infinite-dimensional problem, and we certainly do not claim to prove (nonlinear) stability without the use of functional analysis.

THEOREM 4.15. *Suppose that $\lambda > \lambda_0$. Choose r_1 with $0 < r_1 < \frac{1}{2}$. Let*

$$\mu = \int_{(1-r_1)\pi}^{(1-\frac{r_1}{2})\pi} \sin s\, ds.$$

Choose $\delta > 0$ so that

(i) $\delta < \underline{U}\left(\frac{r_1}{2}\pi\right) - \underline{U}(0)$;

(ii) *if $\bar{U}(t) - \delta < u(t) < \bar{U}(t)$ for some t, then $|u^3 - \lambda u + \cos t| < \mu$.*

Let S_δ denote the set of points (t,u) such that $0\leqslant t\leqslant\pi$, $-b\leqslant u\leqslant b$, and

$$\begin{aligned} -b\leqslant u\leqslant \underline{U}(t)+\delta \qquad & \text{if } 0\leqslant t\leqslant r_1\pi, \\ \bar{U}(t)-\delta\leqslant u\leqslant b \qquad & \text{if } (1-r_1)\pi\leqslant t\leqslant\pi. \end{aligned}$$

Then for sufficiently small ε, u_p is the only 2π-periodic solution of (1.4) *with $u'(0)=u'(\pi)=0$ whose graph over $[0,\pi]$ lies in S_δ. Further, $u_\mathrm{p}|_{[0,\pi]}$ is a stable attractor for the problem* (1.2)–(1.3) *with $L=\pi$.*

Also, suppose that S_δ is extended to $[0,2\pi]$ by the operation $(t,u)\to(2\pi-t,u)$, and then periodically to $[0,\infty)$. Then for any positive integer k, u_p is the unique solution of (1.4) *satisfying $u'(0)=u'(k\pi)=0$ whose graph lies in the extended set S_δ, and its restriction to $[0,k\pi]$ is a stable attractor for* (1.2)–(1.3) *with $L=k\pi$.*

To compare this with the result in [AMPP] we notice that the width of the vertical strip in S_δ where the solution can increase from near $\underline{U}$ to near $\bar{U}$ is $(1-2r_1)\pi$, and this is free to be chosen within the constraint $0<r_1<\frac{1}{2}$. By contrast, [AMPP] state only that there is some strip, with width independent of ε, which contains the "internal layer" of the solution we have denoted by u_p, and no other solution satisfying the boundary conditions has a jump upward within this strip.

The proof in [AMPP] is by a detailed construction of u_p using sub- and super-solutions. It also uses abstract results from dynamical systems (to get the uniqueness.) Our proof of uniqueness is more direct, and starts with the existence of u_p as given in Theorem 2.6.

Proof. To prove Theorem 4.15 we consider the variational equation and initial conditions satisfied by $v=\frac{\partial u_\alpha}{\partial\alpha}$. These are

$$\begin{aligned} &\varepsilon^2 v''=(3u^2-\lambda)v, \\ &v(0)=1, \qquad v'(0)=0. \end{aligned} \tag{4.19}$$

We will also be concerned with $w=u'$, which satisfies

$$\begin{aligned} &\varepsilon^2 w''=(3u^2-\lambda)w-\sin t, \\ &w(0)=0, \qquad w'(0)=u''(0). \end{aligned} \tag{4.20}$$

We observe that $w'(0)>0$ when $u(0)\in(\underline{U}(0),0)$. Multiplying (4.19) by w and (4.20) by v, subtracting, integrating by parts and using the initial conditions on v and w, we obtain

$$wv'-vw'|_t=-u''(0)+\int_0^t\frac{v(s)}{\varepsilon^2}\sin s\,ds. \tag{4.21}$$

LEMMA 4.16. *If u is a solution with $u'(0) = 0$ which remains in S_δ on $[0, \pi]$, then $v > 0$ on $[0, \pi]$ and $v'(\pi) > 0$.*

Proof. Because $u < \underline{U}(t) + \delta$ over the interval $[0, r_1\pi]$, and therefore $3u^2 - \lambda > 0$ in this interval, (4.19) implies that v grows exponentially large. More precisely, there are positive numbers $K_1 \geqslant 1$ and γ, independent of ε, such that $v \geqslant K_1 e^{\frac{\gamma}{\varepsilon}t}$ in $[0, r_1\pi]$. Also, $u''(0) = O\left(\frac{1}{\varepsilon^2}\right)$ as $\varepsilon \to 0$. Therefore, for small ε, the right side of (4.21) is positive as long after $t = r_1\pi$ as v is positive (up to $t = \pi$).

We now show that $u'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) > 0$. If not, then $u' < 0$ on $\left(\left(1 - \frac{r_1}{2}\right)\pi, \pi\right)$, since $u'' < 0$ when $\bar{U} - \delta < u < \bar{U}$, and $\bar{U}' > 0$ in $\left(\frac{\pi}{2}, \pi\right)$. Therefore, $u(\pi) < u\left(\left(1 - \frac{r_1}{2}\right)\pi\right)$. From (i) it follows that $u(\pi) < \bar{U}(\pi) - \delta$, contradicting the assumption that u remains in S_δ. Hence, $u'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) > 0$. This implies that $u' > 0$ on $(0, (1 - \frac{r_1}{2})\pi]$, for if not, then u would have a minimum in this interval, and this minimum would lie above $u(0)$, contradicting Lemma 4.10.

Now suppose in (4.21) that $v = 0$ somewhere in $(0, (1 - \frac{r_1}{2})\pi]$. Then the right side of (4.21) is positive, while the left side is negative. Hence, $v > 0$ on $[0, (1 - \frac{r_1}{2})\pi]$.

It is possible that v' becomes negative somewhere in $[0, \pi]$. Indeed, numerically this is seen to happen. However, we will show that $v'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) > 0$. Suppose that $v'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) \leqslant 0$. Then $v'((1 - r_1)\pi) < 0$, for otherwise v' would be positive and increasing on $[(1 - r_1)\pi, (1 - \frac{r_1}{2})\pi)$, because $(3u^2 - \lambda) > 0$ there. Thus, v is positive but decreasing on $[(1 - r_1)\pi, \left(1 - \frac{r_1}{2}\right)\pi]$. From (4.21) we obtain

$$\begin{aligned}
&v'\left(\left(1 - \frac{r_1}{2}\right)\pi\right)u'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) - v\left(\left(1 - \frac{r_1}{2}\right)\pi\right)u''\left(\left(1 - \frac{r_1}{2}\right)\pi\right) \\
&\quad = \; - u''(0) + \int_0^{(1-r_1)\pi} \frac{v(s)}{\varepsilon^2} \sin s \, ds + \int_{(1-r_1)\pi}^{\left(1-\frac{r_1}{2}\right)\pi} \frac{v(s)}{\varepsilon^2} \sin s \, ds \\
&\quad \geqslant v\left(\left(1 - \frac{r_1}{2}\right)\pi\right) \int_{(1-r_1)\pi}^{\left(1-\frac{r_1}{2}\right)\pi} \frac{\sin s}{\varepsilon^2} \, ds.
\end{aligned}$$

Since $u'\left(\left(1 - \frac{r_1}{2}\right)\pi\right) > 0$, we get a contradiction from condition (ii) in the statement of the theorem, and so v' could not remain negative on $\left[(1 - r_1)\pi, \left(1 - \frac{r_1}{2}\right)\pi\right]$. Therefore, at $\left(1 - \frac{r_1}{2}\right)\pi$ we have both v and $v' > 0$, and since $3u^2 - \lambda > 0$ on $\left[\left(1 - \frac{r_1}{2}\right)\pi, \pi\right]$, both remain positive out to π. This proves the lemma. ∎

In particular, this applies to the solution u_{p}. It follows by standard stability arguments [BF] that the solution u_{p} is a stable attractor for the problem (1.2)–(1.3) on $[0, \pi]$.

To prove that u_p is the only 2π-periodic solution which remains in S_δ, it is convenient to truncate the nonlinearity in (1.4) by replacing $u^3 - \lambda u$ by $b^3 - \lambda b$ for all $u \geqslant b$, and similarly, if $u \leqslant -b$, replace $u^3 - \lambda u$ in (1.4) with $\lambda b - b^3$. This means that all solutions u_α exist on $[0, \pi]$, and we can consider $u'_\alpha(\pi)$ to be defined continuously for all negative α. We also note that if u_α is a solution which does leave the region $[-b, b]$, then from the point where $|u_\alpha| = b$, $|u'_\alpha|$ continues to increase and cannot satisfy $u'_\alpha = 0$.

Recall that u_1 is the only 2π-periodic solution of (1.4) lying entirely below $-\sqrt{\frac{\lambda}{3}}$; this is the "minimal" bounded solution, and its graph does not lie in S_δ. In considering the possibility of a second 2π-periodic solution, besides u_p, which lies in S_δ, we need only consider $\alpha \in (\alpha_1, \underline{U}(0) + \delta)$.

The solution u_p remains in the interior of S_δ on $[0, \pi]$, and the same is true for u_α if $|\alpha - \alpha_p|$ is sufficiently small. However, as we raise or lower α from α_p, we reach values where u_α leaves S_δ in $[0, \pi]$. Let $I = [\hat{\beta}, \hat{\alpha}]$ be the maximal interval containing α_p such that u_α remains in S_δ on $[0, \pi]$ for all $\alpha \in I$. I is well-defined and closed because S_δ is a closed set.

Lemma 4.17. *If $\alpha \notin I$, then u_α does not remain in S_δ on* $[0, \pi]$.

Proof. We claim that $u_{\hat{\beta}}$ leaves S_δ at $(\pi, \bar{U}(\pi) - \delta)$ and $u_{\hat{\alpha}}$ exits S_δ at (π, b). If not, then one of these solutions is tangent to the boundary of S_δ at some $t < \pi$. For example, a tangency could occur at $((1 - r_1)\pi, \bar{U}((1 - r_1)\pi))$. But then, no matter what the slope of u is at this point, a phase plane argument shows that for sufficiently small ε, u must cross $-b$ in one direction or the other, and so nearby solutions also leave S_δ before $t = \pi$, contradicting the definition of $\hat{\alpha}$ or $\hat{\beta}$. Similar considerations apply at any other possible tangent point in $[0, \pi)$.

Consider the case of $u_{\hat{\beta}}$. By Lemma 4.16, $\frac{\partial}{\partial\alpha} u_\alpha(\pi)|_{\alpha=\hat{\beta}} > 0$. Hence, for $\alpha < \hat{\beta}$ and close to $\hat{\beta}$, $u_\alpha(\pi) < \bar{U}(\pi) - \delta$, and we cannot, as we lower α, find a lower α where u_α remains in S_δ and $u_\alpha(\pi) = \bar{U}(\pi) - \delta$, for at the first such point we would have $\frac{\partial}{\partial\alpha} u_\alpha(\pi) \leqslant 0$, a contradiction to Lemma 4.16. Similar remarks apply to $u_{\hat{\alpha}}$, completing the proof of Lemma 4.17. ∎

The proof that if $u = u_p$, then $v > 0$ on $[0, \pi]$ and $v'(\pi) > 0$, shows that in some neighborhood of $\alpha_p, (\alpha - \alpha_p)u'_\alpha(\pi) > 0$. Now, if there is an $\check{\alpha} \in (\alpha_p, \hat{\alpha})$ with $u'_{\check{\alpha}}(\pi) = 0$, then choose the smallest such $\check{\alpha}$. By Lemma 4.16 we get $\frac{\partial}{\partial\alpha} u'_\alpha(\pi) > 0$ at α_p and $\check{\alpha}$, a contradiction because these are adjacent zeros of $u'_\alpha(\pi)$. This contradiction can be reached similarly in the interval $(\hat{\beta}, \alpha_p)$, completing the proof of the uniqueness of u_p among solutions with period 2π which remain in S_δ.

To extend the uniqueness and stability statements to larger intervals we note that starting with $v(\pi) > 0$, $v'(\pi) > 0$, the same analysis allows us to show inductively that v remains positive, and v' is positive at any multiple of

π. The arguments about uniqueness and stability can then be extended to $[0, k\pi]$ completing the proof of Theorem 4.15. ∎

4.4. Sensitivity with Respect to Initial Conditions

As this paper is already quite long, we will content ourselves with a few remarks. Up until now there has been no mention of "horseshoes" in this paper, or of Poincaré maps, because our technique is to follow complete solutions of the ode, rather than to take snapshots at regular intervals. The results, however, are related to standard dynamical systems concepts such as horseshoes and sensitivity to initial conditions.

In Theorems 3.2 and 3.5, we obtain a weak kind of sensitivity to initial conditions. For each sequence there is a corresponding solution, but the relation established is not 1:1. There could be an interval of initial values α in which u_α intersects the same sequence of w_k. This corresponds to a so-called "topological" horseshoe, without the hyperbolicity that was a key feature of Smale's original derivation. (See [GH] for general discussion and references.) It is noted, however, that the solutions in Theorem 3.2 corresponding to the sequences of all odd integers and of all even integers, are our periodic solutions u_1 and u_5, and for these, uniqueness of the correspondence is established. The difficulty is the stability, or at least hyperbolicity, of solutions u_2, u_3, and u_4.

In Section 4.3, hyperbolicity was established for u_3 when ε is sufficiently small. Theorem 4.15 implies that any solution which corresponds to a sequence with no 2 or 4 is isolated from any other such solution. Therefore, in Theorem 4.15, if we consider a sequence with no 2 or 4, then the infinite intersection of closed intervals used in the construction contains exactly one point. This means that the set of solutions found in Theorem 3.2 corresponding to sequences chosen from the set $\{1, 3, 5\}$ is, for sufficiently small ε, in 1:1 correspondence with the set of allowed symbol sequences. (The rule that 1 and 5 cannot follow each other must still be obeyed.) Hence, the desired degree of sensitivity to initial conditions, in which any small perturbation of the initial condition leads to a deviation from the given sequence, is achieved.

In the paper [HM2], a similar result was obtained. There, however, the analysis was not only for stable solutions. Indeed, only one of all the solutions found in that paper is stable in the linearized sense. However they are all hyperbolic. We believe that a similar analysis will allow us to study linearizations around solutions u_2 and u_4 and prove their isolation as well. However, we will not attempt this analysis here.

4.5. Bifurcation in λ

We saw in the Introduction that at some $\lambda_b \in (0, \lambda_0]$ new periodic solutions appear. While we know that for $\lambda > \lambda_0$ there are at least five solutions with period 2π, we have not discussed the nature of the bifurcation. It is possible that the number of solutions goes from one to three at λ_b and later increases to five. Stability considerations do not rule this out.

One way of studying this numerically is to consider the graph of $G(\alpha) = u'_{\alpha}(\pi)$, since we are only considering periodic solutions with the properties $u'(0) = u'(\pi) = 0$. If the bifurcation is "pitchfork," then the graph of G will qualitatively resemble that of the function $\alpha^3 - \mu\alpha$ as the parameter μ changes from negative to positive. (The zeros of G will not be at $\alpha = 0$.) However, a numerical study of the function G quickly suggests that this is not the case. In Fig. 7, we show the resulting graph of G for $\varepsilon = 1$ close to the bifurcation point, $\lambda = 1.023$. This indicates that two pairs of solutions bifurcate at the same value of λ. This is partly a trivial observation, however, for the symmetry in the problem shows that solutions other than those for which $u\left(\frac{\pi}{2}\right) = 0$ occur in symmetric pairs. The essential nature of the bifurcation can be seen by looking only at the left branch of the bifurcation curve shown in Fig. 7.

To study the bifurcation analytically, we again use $v = \frac{\partial u}{\partial \alpha}$. We are considering only the specific equation (1.4). In order to prove that the bifurcation is not pitchfork, we study the linearization around the antisymmetric solution u_{p}. We determine the slope of the function G at

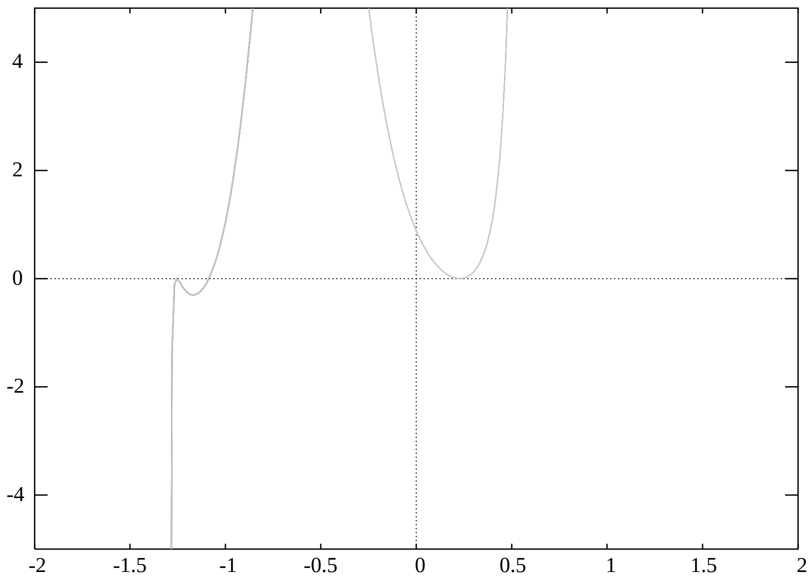

FIGURE 7

the antisymmetric solution. Since $G(\alpha) = u'_\alpha(\pi)$, $G'(\alpha) = v'_\alpha(\pi)$. But in the proof of Theorem 4.15 we showed that when $u = u_{\mathrm{p}}$, $v > 0$ on $[0, \pi]$ and $v'(\pi) > 0$. This proves that the bifurcation is not of pitchfork type, because the new solutions must appear at a positive distance away from u_{p}.

However, this analysis does not eliminate the possibility of several bifurcation points, in which solutions appear and then disappear. To get a complete picture, more work is required. We have seen that for $\lambda > \lambda_0$ there are at least five solutions. We also showed that λ_b, the bifurcation point, is less than λ_0. But in the following result, we hold λ fixed and less than λ_0, and let ε tend to zero.

THEOREM 4.18. *Suppose that $\lambda < \lambda_0$. Then for sufficiently small ε, u_{p} is the only solution with $u'(0) = 0$, $u'(\pi) = 0$.*

COROLLARY 4.19. $\lim_{\varepsilon \to 0} \lambda_b = \lambda_0$.

Proof. The corollary follows immediately from the theorem, the proof of which is more easily understood by reference to Fig. 1. We use the original scaling (1.4). For any $\delta > 0$, there are $\mu > 0$ and $v > 0$ such that $f(t, u) \leqslant -v$ on the set

$$\Omega_{\delta,\mu} = \{(t, u) \mid (1 - \mu)\pi \leqslant t \leqslant (1 + \mu)\pi, \ u \leqslant \bar{U}(t) - \delta\}.$$

Hence, for sufficiently small ε, no periodic solution can intersect the region $\Omega_{\delta,\mu}$ at a point where $\left(1 - \frac{\mu}{2}\right)\pi \leqslant t \leqslant \left(1 + \frac{\mu}{2}\right)\pi$, since then u would be forced below $-b$ before $|t - \pi| = \mu$. So any periodic solution must have lie above $\bar{U}(t) - \delta$ on $\left(1 - \frac{\mu}{2}\right)\pi \leqslant t \leqslant \left(1 + \frac{\mu}{2}\right)\pi$. The argument used to prove Theorem 4.15 can then be used to show that for sufficiently small ε, u_{p} is the only periodic solution u_α with a maximum at π. This completes the proof of the theorem and corollary. ∎

In studying the bifurcation we now look at a simpler problem, where we study monotone solutions. This eliminates the layer-type solutions of Section 4.2, which may also exist near the bifurcation point. So, let $\lambda_{mb} = \sup\{\lambda \mid \text{there is only one solution which is monotone on } [0, \pi]\}$. We now show that for small ε at least four new periodic solutions appear (i.e. two symmetric pairs), and there is no "reverse bifurcation" as λ increases from λ_{mb}. These four new solutions are monotone increasing on $[0, \pi]$. In other words, as we increase λ from λ_{mb}, there is no return to the case where u_{p} is the unique $2 - \pi$ periodic solution which is monotone on $[0, \pi]$. We now explicitly show the dependence of λ_{mb} on ε.

THEOREM 4.20 *For sufficiently small $\varepsilon > 0$, if $\lambda > \lambda_{mb}(\varepsilon)$, then there are at least five periodic solutions, $u_1, \ldots, u_5$ with $\alpha_i = u_i(0) < 0$. Further, $u_i < u_{i+1}$ for*

$i = 1, \ldots, 4$. *Here*, u_3 *is the antisymmetric solution*, *also denoted by* u_{p}. *The solutions* u_1 *and* u_5 *are symmetric reflections*, *with* $u_5(t) = -u_1(t + \pi)$. *Similarly*, $u_4(t) = -u_2(t + \pi)$.

Proof. We have seen that $G(\alpha) < 0$ for large negative α and also $G < 0$ just below α_{p}. Let

$$\alpha_1 = \alpha_1(\lambda, \varepsilon) = \inf\{\alpha |\ u'_\alpha(\pi) = 0\}.$$

At any bifurcation point where the transition is from one solution to more than one solution, or vice versa, $G(\alpha_1) = G'(\alpha_1) = 0$. The existence of at least five solutions for λ just above λ_{mb} follows from the remarks above by showing that $\frac{\partial G(\alpha_1(\lambda_{mb},\varepsilon))}{\partial \lambda} > 0$. Let $u = u_{\alpha_1(\lambda_{mb},\varepsilon)}$, $h(t) = \frac{\partial u_{\alpha_1(\lambda_{mb},\varepsilon)}(t)}{\partial \lambda}$, and $v = \frac{\partial u}{\partial \alpha}|_{\alpha=\alpha_1(\lambda_{mb},\varepsilon)}$. We must show that $h'(\pi) > 0$. We see that

$$\begin{aligned} \varepsilon^2 h'' &= (3u^2 - \lambda)h - u, \\ h(0) &= h'(0) = 0 \end{aligned} \tag{4.22}$$

while v satisfies (4.19). Therefore we obtain, for any $t \in [0, \pi]$,

$$hv' - vh'|_t = \frac{1}{\varepsilon^2} \int_0^t v(s)u(s)\, ds. \tag{4.23}$$

Lemma 4.21. *For sufficiently small* ε, $u_{\alpha_1(\lambda_{mb},\varepsilon)} < 0$ *on* $[0, \pi]$.

Proof. This follows by a modification of the proof of Theorem 4.18. Let

$$\hat{\Omega}_{\delta,\mu} = \{(t, u) \,|\, (1 - \mu)\pi \leqslant t \leqslant (1 + \mu)\pi,\ 0 \leqslant u \leqslant \bar{U}(t) - \delta\}.$$

Suppose that $\lambda = \lambda_0$. Then for small enough δ and μ, there is an ε_0 such that if $0 < \varepsilon \leqslant \varepsilon_0$, then any solution which intersects $\hat{\Omega}_{\delta,\mu}$ must decrease monotonically, in at least one direction, to below $-b$, within a time which is bounded over $\hat{\Omega}_{\delta,\mu}$. Corollary 4.19 implies that ε_0 can also be chosen so that this conclusion is also true if $\lambda_{mb}(\varepsilon) \leqslant \lambda < \lambda_0$, for a fixed pair (δ, η) independent of λ. As in the proof of Theorem 4.15, no solution other than u_{p} can have a positive maximum at π and not intersect $\hat{\Omega}_{\delta,\mu}$, proving the result. ∎

Also, if $u = u_{\alpha_1(\lambda_{mb},\varepsilon)}$, then $v'(\pi) = G'(\alpha_1) = 0$. Therefore $v(\pi) \neq 0$. Suppose that $v(\pi) < 0$. Then for α slightly lower than $\alpha_1 = \alpha_1(\lambda_{mb}, \varepsilon)$, $u_\alpha(\pi) > u_{\alpha_1}(\pi)$. Let

$$\beta = \inf\{\alpha \,|\, u_\alpha(t) > u_{\alpha_1}(t) \text{ for some } t \in (0, \pi]\}.$$

Then β is well defined and $-\infty < \beta < \alpha_1$. Suppose that $u_\beta(t_0) = u_{\alpha_1}(t_0)$ for some $t_0 \in (0, \pi)$. Since different solutions cannot be tangent, we must have $u_\beta(t) > u_{\alpha_1}(t)$ for some $t \in (0, \pi)$, but this contradicts the definition of β. Therefore, $u_\beta(\pi) = u_{\alpha_1}(\pi)$ and $u'_\beta(\pi) > u_{\alpha_1}(\pi) = 0$. But in this case, we can lower α further, until we find a $\gamma < \beta$ with $u'_\gamma(\pi) = 0$. This contradicts the definition of α_1. A similar argument shows that $v > 0$ on $[0, \pi]$.

Therefore, when $u = u_{\alpha_1}$, $v(\pi) > 0$. Then (4.23) and Lemma 4.21 show that $\frac{\partial u'_{\alpha_1(\lambda_{mb},\varepsilon)}(\pi)}{\partial \lambda} = h'(\pi) > 0$. Thus, for λ just above λ_{mb} there are at least five solutions (using symmetry). Further, there cannot be a decrease to fewer than five as λ increases further, for at any point where $G = G' = 0$ we would again get $h'(\pi) > 0$.

To complete the proof of Theorem 4.20, it is convenient to let $\alpha_2 = \sup\{\alpha < \alpha_{\mathrm{p}} \mid u'_\alpha(\pi) = 0\}$. (Numerically, it appears there is only one 2π-periodic solution between u_1 and u_{p}.) Then reflection and translation of $u_{\alpha_1}$and u_{α_2} by the transformation $u(t) \to -u(t + \pi)$ give the additional two asymmetric solutions.

Our construction implies that $u_i(0) < u_{i+1}(0)$. From Proposition 2.4 and the way we define α_2 we have $u'_\alpha(\pi) < 0$ for $\alpha_2 < \alpha < \alpha_{\mathrm{p}}$, and in this range and close enough to α_{p}, $u_\alpha < u_{\mathrm{p}}$. (This is because when $u = u_{\mathrm{p}}$, $v > 0$ on $[0, \pi]$.) Suppose, however, that for some $\alpha \in [\alpha_2, \alpha_{\mathrm{p}})$, $u_\alpha(t) \geqslant u_{\mathrm{p}}(t)$ for some $t \in [0, \pi]$. Let $\breve{\alpha} = \sup\{\alpha \in [\alpha_2, \alpha_{\mathrm{p}}) | u_\alpha(t) = u_{\mathrm{p}}(t)$ for some $t \in [0, \pi]\}$. By the same argument as above we show that $u_{\breve{\alpha}}(\pi) = u_{\mathrm{p}}(\pi)$ and $u'_{\breve{\alpha}}(\pi) > 0$. Hence, there is an $\alpha \in (\breve{\alpha}, \alpha_{\mathrm{p}})$ with $u'_\alpha(\pi) = 0$. But this contradicts the definition of α_2.

This proves that $u_2 < u_{\mathrm{p}}$. The proof that $u_1 < u_2$ is similar, and the construction of u_4 and u_5 by reflection and translation implies the remaining order relations, namely, $u_{\mathrm{p}} = u_3 < u_4 < u_5$. This completes the proof of Theorem 4.20. ∎

5. CONCLUSION

5.1. How Special Is the Cosine?

General nonsymmetric forcing functions are a topic for further study, but a few things are easy to see. First, the procedure in Theorems 3.2 and 3.5 for obtaining some sort of chaotic behavior will carry over to a large variety of forcing functions. Also, on a finite interval the technique will give many steady states for problem (1.2)–(1.3). The only requirement is the existence of some set of functions w_k whose graphs form "fingers" pointing up and down alternately in a way similar to that in Fig. 2. We will not try to formulate a precise result here.

We did use symmetry essentially to obtain the existence of the solution u_{p}. It is here that we have found significant differences between the problem of finding steady states for (1.2)–(1.3) on a finite interval and the problem of finding bounded solutions, and chaos, on an infinite interval. It is easy to find a third solution to the boundary value problem on a finite interval using shooting. From there one can go on to find many other solutions, both stable and unstable. But dealing with chaos on an infinite interval seems different, and it is for that reason that we concentrated on the particular equation (1.4). Having done so, it is natural to make use of symmetry to obtain simpler proofs in some cases where a shooting method may apply even without symmetry.

We note, however, that the proof of Theorem 3.2, which does not use symmetry, includes the existence of a solution which does not intersect any of the w_k. This solution is the stable solution of [AMPP], and could play the role of u_{p} in a study which uses shooting but does not use symmetry.

Turning to the small ε results, we believe that if the function g has a positive local minimum or negative local maximum, then the energy argument Lemma 4.10 leads to some new solutions, namely multiple spikes at these new types of critical points of g. Regarding the types of solutions found in this paper, the proofs in [AMPP] make no use of symmetry, so the stable solutions found there continue to exist in its absence. We conjecture that the unstable solutions we have found, namely the single unstable spikes like u_2 and u_4, and the multiple layer solutions at zeros of g, as in Theorem 4.12 also persist, and can be found by our methods.

The results about bifurcation as λ increases, however, may change radically if g has less symmetry. For example, if g has period 2π, and is symmetric around π, but not antisymmetric around $\frac{\pi}{2}$, then the solutions will not appear in symmetric pairs. The technique we have used may still be able to prove that saddle-node bifurcations occur, at least in a nearly-symmetric situation, but there may be two of them, producing first two new solutions and then two more. This is easily seen in numerical simulations. As the deviation from full symmetry increases, it appears from some brief numerical experiments that other possibilities exist, and we hope to explore this further.

5.2. Summary of Main Points

In Section 1, we introduce the problem and relate it to some previous work.

In Section 2, we give preliminary results valid for all $\varepsilon > 0$. Some of these are for a more general forcing function $g(t)$. The main points are that for $\lambda \leqslant 0$ there is a unique bounded solution, which is periodic, while for $\lambda > \lambda_0$ there are at least three solutions. Therefore a bifurcation takes place, and some preliminary computations have suggested that this can be of different types depending on $g(t)$.

In Section 3, we give results valid for a specific range of ε and λ. The main hypothesis is Condition 3.1, the existence of "spikes" (the w_k), which are solutions tending to $\pm\infty$ in both directions. Solutions are characterized by which of the w_k they intersect. Section 3.1 gives the proof of the 1:1 correspondence with certain sequences if Condition 3.1 holds. It would be possible to rephrase this result to give a natural correspondence with sequences of three symbols, corresponding to the three solutions which we later labeled $u_1, u_3 (= u_{\mathrm{p}})$, and u_5. In Section 3.2, we give a brief discussion of "kneading theory" in our context. In Section 3.3, the symbolic dynamics is extended to sequences of five symbols. In Section 3.4, Condition 3.1 is verified, first for "sufficiently small" ε, where no analysis is required, and then for larger ε.

The results in Section 3 do not include uniqueness or stability, and there is only a limited sensitivity to initial conditions demonstrated. They do not depend at all on symmetry, and indeed, the techniques will yield a weak form of chaos for a wide variety of forcing functions, including nonperiodic forcing.

Section 4 contains a variety of results, all proved for sufficiently small ε with no estimate on the range of ε for which they hold. In Section 4.1 some results are given about asymptotic behavior of solutions as $\varepsilon \to 0$. In Section 4.2, further periodic solutions are found, including solutions with multiple internal layers and with a "down-jump" near $\frac{\pi}{2}$, in contrast to the solution u_{p} which jumps upward at $\frac{\pi}{2}$. In Section 4.3, we give a proof of the stability of u_{p} using classical ode methods, and extend the result on uniqueness of [AMPP] a bit by obtaining a larger region in which it is unique. The uniqueness proof is also more direct than that in [AMPP].

In Section 4.4, we discuss sensitivity with respect to initial conditions. In Section 4.5, we consider the bifurcation problem in λ for 2π-periodic solutions of (1.4).

Finally, in Section 5.1, we discuss the role of the specific cosine forcing function in our results, and conjecture that its symmetry and monotonicity properties may not be essential except in the bifurcation analysis of Section 4.6.

We express our appreciation to Professor H. Matano for sharing his thoughts on this problem. His remarks are cited in more detail at the end of Section 4.2, together with a citation of recent work of his co-worker, Dr. K. Nakashima.

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